A NOTE ON THE EQUILIBRIUM POINT OF THE GREEN'S FUNCTION FOR AN ANNULUS

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1. Introduction. In a previous paper* the motion of the equilibrium point of the Green's function for a plane annular region was studied as the pole was shifted along a radius in the neighborhood of the geometric mean circle $C_0$.† The expression for $\frac{dr}{dr_0}$ on $C_0$, $r$ being the distance of the equilibrium point from the center of the circles, $r_0$ that of the pole, is $-\frac{F_{r_0}}{F_r}$, where

$$F_{r_0} = \frac{\partial F}{\partial r_0} = -\frac{2}{R} \left[ \frac{1}{2 \log R} - \frac{1}{8} + \sum_{m=1}^{\infty} \frac{(-1)^m}{R^m - 1} \right],$$

$$F_r = \frac{\partial F}{\partial r} = -\frac{2}{8} \left[ \frac{1}{R} + \sum_{m=1}^{\infty} \frac{(-1)^m}{R^m + 1} \right].$$

In these formulas $1$ and $R$ are the radii of the inner and outer circular boundaries of the region. It was shown by an application of a theorem of Schlömilch‡ that $F_{r_0}$ does not vanish on $C_0$.

In this article this result and others are obtained by a method which seems better adapted to the problem.§

It is noticed that the function

$$f(z) = \frac{\pi}{\sin \pi a} \frac{z}{e^{az} - 1}, \quad a = \log R,$$

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† The Green's function for this region may be written in the form

$$g(M, M_0) = \log \frac{1}{MM_0} + \frac{1}{\log R} \left[ \log R \log r_0 - \log r \log r_0/R \right]$$

$$- \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cos m(\theta - \theta_0)}{R^{2m} - 1} \left( r^m[r_0^m - r_0^{-m}] + r^{-m} \left( \frac{r_0^2}{r_0^2} \right)^m - r_0^m \right).$$

We take $F(r, r_0) = \frac{\partial g}{\partial r}$ for $r = r_0 = R^{1/2}$ and $\theta - \theta_0 = \pi$.
§ The suggestion that the method of contour integration and the theory of residues might prove useful was given by A. J. Maria.
where $R$ is a real number greater than 1, has the sum of residues $\sum_{m=1}^{n} (-1)^{m}m/(R^m - 1)$ within a suitably selected contour containing as singularities only the poles 1, 2, $\cdots$, $n$ of $f(z)$. It turns out by an integration around a contour and certain limiting processes that an expression for $F_r$, on $C_0$ is found which is a series of positive terms. The same method applied to a suitably chosen function $f(z)$ yields for the value of $F_r$ a series of negative terms. These values show that $dr/dr_0$ is positive on $C_0$. By means of the preceding results it is proved that $d^2r/dr_0^2$ is negative on $C_0$.*

2. The Evaluation of $F_{rQ}$ and $F_r$ on $C_0$ by Contour Integration.

The contour $C_n$ chosen for the evaluation of $F_{rQ}$ consists of the lines $x_n = n + 1/2$, $y_n = \pm (2n + 1)\pi/a$, semi-circular arcs that lie to the right of the imaginary axis of positive radius $\rho < \pi/(2a)$ and $<1$ and centers $\pm 2mn\pi/a$, ($m = 0, 1, \cdots, n$), and the straight line segments of the imaginary axis exterior to these arcs included between the upper and lower $y_n$ lines. The function

$$f(z) = \frac{\pi z}{\sin \pi z e^{az} - 1}$$

is analytic inside and on $C_n$ except at the poles $z = 1, 2, \cdots, n$ of $\pi/\sin \pi z$. Hence the value of the integral $(1/2\pi i) \int_{C_n} f(z) dz$, where the contour $C_n$ is traced in the counter-clockwise direction, gives $\sum_{m=1}^{n} (-1)^{m}m/(e^{am} - 1)$, the sum of the residues of $f(z)$ inside $C_n$.

Let $L_n$ be the straight line segments on the imaginary axis, $K_n$ the semi-circular arcs, and $S_n$ the remaining part of $C_n$. Over $L_n$ the integral can easily be put into the form

$$- \frac{1}{2} \sum_{m=0}^{n-1} \int_{(2m+1)\pi/a-\rho}^{(2m+1)\pi/a+\rho} \frac{y dy}{\sin \pi y} - \frac{1}{2} \int_{(2n+1)\pi/a}^{(2n+1)\pi/a+\rho} \frac{y dy}{\sin \pi y}.$$

For the evaluation of the integral over the arc of $K_n$ with center at $2m\pi/a$, ($m \neq 0$), a power series development of $f(z)$ about this point is used. Evaluated, the integral gives

$$- \frac{\pi^2}{a^2} \frac{m}{\sinh (2m\pi^2/a)} + P_m(\rho),$$

* It is evident that the corresponding results hold for any annulus.
where $P_m(\rho)$ is a power series in $\rho$ with constant term zero. Over the arc with center at the origin the value is found to be $(-1/(2a)) + P_0(\rho)$. Thus integration around $C_n$ gives

$$
\sum_{m=1}^{n} \frac{(-1)^m m}{e^{am} - 1} = \frac{1}{2} \sum_{m=0}^{n-1} \int_{(2m\pi/a)+\rho}^{(2(m+1)\pi/a)-\rho} \frac{ydy}{\sinh \pi y}
$$

(1)

$$
+ \frac{1}{2} \int_{(2n\pi/a)+\rho}^{(2(n+1)\pi/a)-\rho} \frac{ydy}{\sinh \pi y} - \frac{1}{2a} \sum_{m=1}^{n} \frac{m}{\sinh (2m\pi^2/a)} + \sum_{m=-n}^{n} P_m(\rho) + \frac{1}{2\pi i} \int_{S_n} f(z)dz.
$$

The value of $(1/2\pi i)\int_{C_n} f(z)dz$ is clearly the same for any positive $\rho$ less than both $\pi/(2a)$ and 1. Letting $\rho$ approach zero in (1), we obtain

$$
\sum_{m=1}^{n} \frac{(-1)^m m}{e^{am} - 1} = \frac{1}{2} \int_{0}^{(2n+1)\pi/a} \frac{ydy}{\sinh \pi y}
$$

(2)

$$
- \frac{2\pi^2}{a^2} \sum_{m=1}^{n} \frac{m}{\sinh (2m\pi^2/a)} - \frac{1}{2a} + \frac{1}{2\pi i} \int_{S_n} f(z)dz.
$$

Now let $n$ become infinite. The left member of (2) has as limit the convergent series $\sum_{m=1}^{\infty} (-1)^m m/(e^{am} - 1)$. The first term on the right approaches the definite integral

$$
\frac{1}{2} \int_{0}^{\infty} \frac{ydy}{\sinh \pi y},
$$

which is known to have the value $1/8$. The series approaches

$$
- \frac{2\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{m}{\sinh (2m\pi^2/a)}.
$$

The integral over $S_n$ has the limit zero.

To prove this last statement consider the modulus of $\int_{S_n} f(z)dz$. It can be shown* that over the entire curve $S_n$, $|1/\sin \pi z|$ and $|1/(e^{az} - 1)|$ are bounded independently of $n$.

* That $|1/\sin \pi z|$ is bounded on $S_n$ independently of $n$ is proved essentially by Lindelöf in *Théorie des Résidus*, 1905, p. 32, footnote. The statement for $|1/(e^{az} - 1)|$ can be proved in the same manner.
Let $M$ be the greater of these two bounds. Moreover, on the upper and lower $y_n$ lines

$$\left| \frac{1}{\sin \pi z} \right| < \frac{1}{\sinh((2n+1)\pi^2/a)},$$

and on the right-hand boundary $z = (n + 1/2) + iy$,

$$\left| \frac{1}{e^{az} - 1} \right| < \frac{1}{e^{a(n+1/2)} - 1}.$$ 

It then follows easily that

$$\left| \int_{s_n} f(z) dz \right| < \frac{Mk(2n + 1)^2}{2a} \left[ \frac{1}{2 \sinh((2n + 1)\pi^2/a)} + \frac{1}{e^{a(n+1/2)} - 1} \right],$$

where $k$ is a constant independent of $n$. This is sufficient to prove the statement.

In the limit for $n$ infinite, (2) gives

$$(3) \quad \frac{1}{2a} - \frac{1}{8} + \sum_{m=1}^{\infty} \frac{(-1)^m}{e^{am} - 1} = - \frac{2\pi^2}{a^2} \sum_{m=1}^{\infty} \frac{m}{\sinh(2m\pi^2/a)}.$$

The left side of (3), where $a$ is replaced by $\log R$, multiplied by $-2/R$, is $F_{r_0}$. Thus $F_{r_0}$ is positive.

For the evaluation of $F_r$ let

$$\frac{\pi}{\sin \pi z} \frac{z}{e^{az} + 1}$$

be chosen for $f(z)$. Let the contour of integration $C_n$ consist of the lines $x_n = n + 1/2$, $y_n = \pm 2n\pi i/a$, semi-circular arcs to the right of the imaginary axis of radius $\rho < \pi/(2a)$ and with their centers at the points $\pm (2m+1)\pi i/a$, $(m = 0, 1, \cdots, n)$, and the portions of the imaginary axis exterior to these arcs between the upper and lower $y_n$ lines.

Applied to this function over the chosen contour, the method used above yields easily the result

$$(4) \quad \frac{1}{8} + \sum_{m=1}^{\infty} \frac{(-1)^m}{e^{am} + 1} = \frac{\pi^2}{a^2} \sum_{m=0}^{\infty} \frac{2m + 1}{\sinh((2m + 1)\pi^2/a)}.$$
By use of (4) with log \( R = a \), \( F_r \) can be written as

\[
- \frac{2\pi^2}{R(\log R)^2} \sum_{m=1}^{\infty} \frac{(2m + 1)}{\sinh \left( \frac{(2m + 1)\pi^2}{\log R} \right)},
\]

a series of negative terms. With these values for \( F_{r0} \) and \( F_r \), we conclude that \( dr/dr_0 = -F_{r0}/F_r \) on \( C_0 \) is positive.

3. The Sign of \( d^2r/dr_0^2 \) on \( C_0 \). From \( dr/dr_0 = -F_{r0}/F_r \), we calculate the second derivative

\[
\frac{d^2r}{dr_0^2} = F_r^{-1} \left[ 2F_{r0}F_rF_{r\varphi} - F_{r0}^2F_{rr} - F_r^2F_{r\varphi\varphi} \right].
\]

From the general expressions for \( F_{r0} \) and \( F_r \) in terms of \( r, r_0, \) and \( R \), the following relations on \( C_0 \) are found to hold

\[
F_{r\varphi\varphi} = - R^{-1/2}F_{r0}, \quad F_{rr} = - 3R^{-1/2}F_r, \quad F_{r\varphi} = - R^{-1/3}F_{r\varphi}.
\]

A substitution of these values in (5) gives

\[
\frac{d^2r}{dr_0^2} = R^{-1/2}F_{r0}F_r^{-1} [F_{r0} + F_r].
\]

Since \( F_{r0} \) and \( F_r \) are positive, the sign of \( d^2r/dr_0^2 \) on \( C_0 \) is that of \( F_{r0} + F_r \). From the results of the preceding section we have

\[
F_{r0} + F_r = \frac{2\pi^2}{R(\log R)^2} \sum_{m=1}^{\infty} \frac{(-1)^m m}{\sinh \left( \frac{m\pi^2}{\log R} \right)}.
\]

This alternating series converges to a negative sum since its terms are in absolute value strictly decreasing to zero. This shows that \( d^2r/dr_0^2 \) is negative on \( C_0 \).