ON A THEOREM OF PLESSNER

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Plessner‡ has shown that if \( f(x) \in L_2 \) on \((-\pi, \pi)\) and

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),
\]

then

\[
\sum_{n=2}^{\infty} (a_n \cos nx + b_n \sin nx)(\log n)^{-1/2}
\]

converges almost everywhere on \((-\pi, \pi)\). We designate the set where (1) converges by \( E(Pl, f) \). This set is then known to be of measure \( 2\pi \). The sets \( E(F, f) \), consisting of the points where

\[
\phi(t) = f(x + t) + f(x - t) - 2f(x) \to 0, \text{ as } t \to 0,
\]

and \( E(L, f) \), consisting of the points where

\[
\Phi(t) = \int_{0}^{t} |\phi(\tau)| \, d\tau = o(t), \text{ as } t \to 0,
\]

are of much importance in the theory of Fourier series. The set \( E(L, f) \) is known to be of measure \( 2\pi \) for all integrable functions. It is obvious that

\[
E(F, t) \subset E(L, f).
\]

We propose in this note to investigate the inclusion relationships between these sets and \( E(Pl, f) \). We shall prove

\[
E(F, f) \subset E(Pl, f),
\]

and

\[
E(Pl, f) \not\subset E(L, f).
\]

We first consider (2). Plessner§ showed that, if (1) converges,

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\[§ \text{Loc. cit., p. 22.}\]
(4) \( S_n(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) = o\{(\log n)^{1/2}\} \).

However, it is well known\(\dagger\) that, for a continuous function, the estimate
\[
(5) \quad S_n(x) = o(\log n)
\]
cannot be improved. This implies that (4) need not be satisfied at every point of continuity and hence
\[
E(F, f) \neq E(\Pi, f).
\]

In order to prove (3) we shall construct a function \( f(x) \in L_2 \) on \((-\pi, \pi)\) for which (1) converges at \( x = 0 \) but such that
\[
(6) \quad \int_{0}^{t} |\phi(\tau)| \, d\tau \neq o(t) \quad \text{as} \quad t \to 0.
\]

The function is similar to one constructed by Paley\(\ddagger\) for another purpose. We define \( f(x) \) by
\[
f(x) = \begin{cases} 
  x \left\{ \left( x - n^{-1} \right) n \log n \right\}^{-1}, & \text{if } n^{-2} \geq |x - n^{-1}| \geq n^{-3}, \\
  0, & \text{elsewhere on } (0, \pi), \\
  f(-x) \text{ for } 0 \geq x \geq -\pi.
\end{cases}
\]

Then, since
\[
\int_{0}^{\pi} |f(x)|^2 \, dx = O \left\{ \sum_{n=3}^{\infty} n^{-4}(\log n)^{-2} \int_{n^{-3}}^{n^{-2}} \frac{dx}{x^2} \right\} = O \left\{ \sum_{n=3}^{\infty} n^{-4}(\log n)^{-2} \right\},
\]
\( f(x) \in L_2 \) on \((-\pi, \pi)\). We have at \( x = 0, \phi(t) = 2f(t) \) and
\[
\int_{0}^{t} |\phi(t)| \, dt > \sum_{n=\lfloor 1/t \rfloor + 1}^{\infty} (n^2 \log n)^{-1} \int_{n^{-2}}^{n^{-1}} \frac{dx}{x} = \sum_{n=\lfloor 1/t \rfloor + 1}^{\infty} n^{-2} > \frac{t}{3}.
\]


for $t$ sufficiently small. Now we consider $S_m(0)$. It is well known that

$$S_m(0) = \frac{1}{2\pi} \int_0^\pi \phi(t) \frac{\sin (m + 1/2)t}{\sin t/2} \, dt$$

$$= \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin mt}{t} \, dt + O(1),$$

while

$$S_m^*(0) = \frac{1}{\pi} \int_0^\pi \phi(t) \frac{\sin mt}{t} \, dt$$

$$= \frac{1}{\pi} \sum_{n=3}^\infty \frac{2}{n \log n} \left\{ \int_{n^{-1}-n^{-3}}^{n^{-1}-n^{-2}} \frac{\sin mt}{(t-n^{-1})} \, dt + \int_{n^{-1}+n^{-3}}^{n^{-1}+n^{-2}} \frac{\sin mt}{(t-n^{-1})} \, dt \right\}.$$ But

$$\sin mt = \sin (mn^{-1}) \cos (m(t-n^{-1}))$$

$$+ \sin (m(t-n^{-1})) \cos (mn^{-1}),$$

so that

$$\pi S_m^*(0) = 4 \sum_{n=3}^\infty (n \log n)^{-1} \cos (mn^{-1}) \int_{n^{-1}}^{n^{-3}} \frac{\sin mt}{t} \, dt$$

$$= 4 \left\{ \sum_{n=3}^{[m^{1/3}]} + \sum_{n=[m^{1/3}] + 1}^{[m^{1/2}]} + \sum_{n=[m^{1/2}] + 1}^\infty \right\}$$

$$= I_1 + I_2 + I_3.$$ Now, if $ma < 1$, $a > b > 0$,

$$\int_b^a \frac{\sin mt}{t} \, dt = \int_{mb}^{ma} \frac{\sin mt}{t} \, dt = O\{ma\},$$

and, if $mb > 1$, $a > b > 0$,

$$\int_b^a \frac{\sin mt}{t} \, dt = \int_{mb}^{ma} \frac{\sin t}{t} \, dt = O\left\{ \frac{1}{mb} \right\} = O(1).$$
Hence
\[ I_1 = O \left\{ \sum_{n=3}^{[m^{1/3}]} (n \log n)^{-1} \frac{n^3}{m} \right\} = O \left\{ \frac{1}{m} \sum_{n=3}^{[m^{1/3}]} \frac{n^2}{\log n} \right\} = o(1), \]
\[ I_2 = O \left\{ \sum_{n=[m^{1/3}]+1}^{[m^{1/3}]} (n \log n)^{-1} \right\} = O \{ \log \log m^{1/2} - \log \log m^{1/3} \} = O(1), \]
\[ I_3 = O \left\{ \sum_{n=[m^{1/3}]+1}^{\infty} (n \log n)^{-1} n^{-3} \right\} = o(1). \]
Therefore
\[ S_m(0) = S_m^*(0) + O(1) = O(1). \]

We now apply Abel's partial summation to (1) and get
\[ \sum_{n=2}^{\infty} (S_n - S_{n-1})(\log n)^{-1/2} \]
\[ = S_1 (\log 2)^{-1/2} + \sum_{n=2}^{\infty} S_n \left\{ (\log n)^{-1/2} - [\log (n + 1)]^{-1/2} \right\}, \]

since \( S_m(0) = O(1) \) and \( (\log n)^{-1/2} \to 0 \) as \( n \to \infty \). But since \( \log n \) is monotone,
\[ \sum_{n=2}^{\infty} \left\{ | (\log n)^{-1/2} - [\log (n + 1)]^{-1/2} | \right\} = (\log 2)^{-1/2}, \]
and therefore (7) converges. This means that the point \( x = 0 \) is contained in \( E(Pl, f) \). But since we have already seen that the point \( x = 0 \) is not contained in \( E(L, f) \), this proves that \( E(Pl, f) \notin E(L, f) \).

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