the connected point sets $B_n$ and $B_m$ have no point in common, there clearly exists no point set consisting of a finite number of connected subsets of $M$ and separating $G$ from $A$ in $M$.

1. Introduction. In this paper the following theorem is proved.

**Theorem.** If $M$ is a plane continuum, and $K$ is a proper subcontinuum of $M$, then at least one component of $M - K$ has a limit point in $K$.

Two points sets are *mutually separated* if they are mutually exclusive and neither of them contains a limit point of the other. A point set is said to be *connected* if it is not the sum of two non-vacuous mutually separated point sets. A point set which is both connected and closed is a *continuum*. A *component* of a point set $N$ is a connected subset of $N$ which is not a proper subset of any other connected subset of $N$. The set of all points in the plane will be denoted by $S$.$\dagger$

2. Proof of the Theorem. If $M$ is a bounded continuum and $K$ is a proper subcontinuum of $M$, it is well known that every component of $M - K$ has a limit point in $K$.‡ If $M$ is unbounded then it is no longer true that every component of $M - K$ has a limit point in $K$.§

If $K$ is a bounded subcontinuum of an unbounded plane continuum $M$, then the above theorem may be proved readily. For

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* Presented to the Society, December 28, 1934. The result of this paper was obtained in 1928, while the author was a student under R. L. Moore at the University of Texas. Recently both R. L. Moore and J. H. Roberts have proved results beyond that of the present paper and have suggested that I publish my original result.

† These definitions are those customarily used in point set theory. See, for example, R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of this Society, vol. 13. For brevity, this treatise will be referred to as "Moore."

‡ See, for example, Moore, p. 24.

§ See Moore, p. 25, example 2,
let $J$ be a simple closed curve enclosing $K$. If $D$ is the interior of $J$ and $M_1$ denotes the component containing $K$ of the subset of $M$ belonging to $D + J$, then $M_1$ is a bounded continuum containing $K$ as a proper subcontinuum. As indicated above, if $C_1$ is a component of $M_1 - K$, then $C_1$ has a limit point in $K$. Hence the component $C$ of $M - K$ which contains $C_1$ has a limit point in $K$, and the theorem is proved.

Now suppose that $K$ is an unbounded proper subcontinuum of the plane continuum $M$. If $x$ is a point of $M - K$, the component of $M - K$ containing $x$ will be denoted by $C_x$. On the assumption that the above theorem is false, it is seen that each component $C_x$ is an unbounded continuum. It follows, therefore, that $M$ is the sum of a set of mutually exclusive unbounded continua consisting of $K$ and the totality of components of $M - K$. This set of mutually exclusive continua is necessarily uncountable in number.*

We shall prove the preceding theorem by showing that the assumption that it is false leads to a contradiction. The proof depends upon the following auxiliary lemma which will be established in the next section.

**Lemma 1.** On the assumption that the above theorem is false, the components $C_x$ of $M - K$ satisfy the following conditions: (a) for each component $C_x$, $M - C_x$ is connected; (b) if $x_1$ and $x_2$ are two points of $M - K$ such that $C_{x_1}$ and $C_{x_2}$ are mutually exclusive, then there does not exist a simple continuous arc $x_1x_2$ such that $x_1x_2 - (x_1 + x_2)$ belongs to $S - M$.

With the help of this lemma the proof is as follows. Let $D$ be a complementary domain of $K$ containing points of $M - K$. Since $M$ is a continuum, it follows that $D$ must contain infinitely many of the components $C_x$ of $M - K$. Let $x_1$ and $x_2$ be points of $M - K$ belonging to $D$ such that $C_{x_1}$ and $C_{x_2}$ are mutually exclusive, and consider a simple continuous arc $x_1x_2$ which belongs to $D$.† Let $X$ denote the subset of points $x$ of $M - K$ such that

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† For a proof that such an arc exists, see R. L. Moore, *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164, in particular, p. 137; see also, Moore, p. 86.
the component $C_x$ has a point in common with the arc $x_1x_2$. Now if $X$ were connected, it would follow that $C_{x_1}$ and $C_{x_2}$ belong to a single component of $M - K$, which is a contradiction. Hence $X$ is the sum of two mutually separated point sets $X_i$ and $X_2$. Clearly, if $x$ is a point of $M - K$ belonging to a set $X_i$, $(i = 1, 2)$, then every point of $C_x$ belongs to the same $X_i$. Let $Y_i$, $(i = 1, 2)$, denote the set of points common to $X_i$ and the arc $x_1x_2$. The sets $Y_i$ are seen to be closed and mutually exclusive. There exists, therefore, a subarc $x_{10}x_{20}$ of $x_1x_2$ such that $x_{10}$ belongs to $Y_i$ and $x_{10}x_{20} - (x_{10} + x_{20})$ belongs to $x_1x_2 - X$ and therefore to $S - M$. This, however, is a contradiction to conclusion $(\beta)$ of Lemma 1, since $x_{10}$ and $x_{20}$ belong to mutually exclusive components of $M - K$. We have proved that the assumption that our theorem is false leads to a contradiction. Hence the theorem is true.

3. **Proof of Lemma 1.** Conclusion $(\alpha)$ of Lemma 1 has been established by Knaster and Kuratowski. We shall prove conclusion $(\beta)$ by showing that the assumption that it is false leads to a contradiction. For suppose that there are two points $x_1$ and $x_2$ of $M - K$ such that $C_{x_1}$ and $C_{x_2}$ are mutually exclusive and there exists a simple continuous arc $x_1x_2$ such that $x_1x_2 - (x_1 + x_2)$ belongs to $S - M$. Let $D$ denote the complementary domain of the continuum $K$ which contains $C_{x_1} + C_{x_2} + x_1x_2$, and denote by $M_1$ the subset of $M$ belonging to $D$. Since $M$ is a continuum, $M_1$ is seen to contain infinitely many of the components $C_x$ of $M - K$. Now let $x'$ be a point of $M_1 - (C_{x_1} + C_{x_2})$, and $P$ a point of the arc $x_1x_2$ between $x_1$ and $x_2$. Since $x'$ and $P$ are both points of $D$, they are not separated by the continuum $K$. Moreover, since $x_1x_2 - (x_1 + x_2)$ belongs to $S - M$ and $M - C_{x_i}$, $(i = 1, 2)$, is connected in view of conclusion $(\alpha)$, there exists a complementary domain $D_i$ of $C_{x_i}$ containing the points $P$ and $x'$, and hence the continuum $C_{x_i}$, $(i = 1, 2)$, does not separate $x'$ and $P$. It follows, therefore, that $K + C_{x_1} + C_{x_2}$ does not separate $x'$ and $P$, and hence there exists a simple continuous arc $x'P$ belong-

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ing to $S - (K + C_{x_1} + C_{x_2})$. Let $O$ denote the first point of the arc $x'P$ belonging to the arc $x_1x_2$, and $x_3$ the last point of the arc $x'O$ belonging to $M$. The point $O$ is seen to be between $x_1$ and $x_2$ on the arc $x_1x_2$, and $x_3$ is a point of $M - K$ such that $C_{x_3}$ is distinct from $C_{x_1}$ and $C_{x_2}$.

Now let $H_j$, ($j = 1, 2, 3$), denote the continuum consisting of the component $C_{x_j}$ and the arc $Ox_j$. Each of the three unbounded continua $H_j$ contains the point $O$, no two of them have in common any point except $O$, and no one of them is separated by the omission of $O$. It then follows* that there is no complementary domain of the continuum $H = H_1 + H_2 + H_3$ whose boundary contains a point, distinct from $O$, of each of the three continua $H_j$. If $D$ denotes the complementary domain of $H$ containing the continuum $K$, the boundary of $D$, aside from the point $O$, belongs to some two of the continua $H_j$. For definiteness, suppose that the boundary of $D$ is a subset of $H_1 + H_2$. Let $D_0$ denote the complementary domain of the continuum $H_1 + H_2$ containing the continuum $C_{x_3}$, and denote by $M_0$ the subset of $M$ belonging to $D_0$. Since $D$ is a complementary domain of $H$, the domain $D_0$ is distinct from the domain $D$ and the point set $M_0$ does not contain $K$. Moreover, since $M$ is a continuum, infinitely many of the components $C_a$ belong to $M_0$.

Now consider the closed point set $N = M_0 + C_{x_1} + C_{x_2}$. If $N$ is connected, we have a contradiction to the assumption that $C_{x_j}$, ($j = 1, 2, 3$), is a component of $M - K$. Suppose, on the other hand, that $N$ is not connected. It will now be shown that under this hypothesis the point set $N$ is the sum of two mutually exclusive closed point sets $N_1$ and $N_2$ such that $N_1$ contains both $C_{x_1}$ and $C_{x_2}$. Since $N$ is closed, the assumption that $N$ is not connected implies that $N = N_{10} + N_{20}$, where $N_{10}$ and $N_{20}$ are mutually exclusive closed point sets. It may be supposed without

* For let $A$ be a point of $S - H$ and subject the plane to an inversion about $A$. If $O$, $H_j$, ($j = 1, 2, 3$), denote the images of $O$, $H_j$, ($j = 1, 2, 3$), under this inversion, then the bounded continua $L_j = H_j + A$ satisfy the following conditions: each of the continua $L_j$ contains the distinct points $A$ and $O$, no two of them have in common any point except $A$ and $O$, and no one of them is separated by the omission of $A + O$. It then follows [see, for example, Moore, p. 291] that there is no complementary domain of the continuum $L = L_1 + L_2 + L_3$ whose boundary contains a point, distinct from $A$ and $O$, of each of the three continua $L_j$, ($j = 1, 2, 3$). Since each complementary domain of $L$ is the image of a complementary domain of $H$, the proof of the above stated result is complete.
loss of generality that $N_{10}$ contains more than one of the components $C_x$ of $M - K$ which belongs to $N$. If $N_{10}$ contains both $C_{x_1}$ and $C_{x_2}$, the property stated above is true for $N_1 = N_{10}$, $N_2 = N_{20}$; if $N_{10}$ contains neither $C_{x_1}$ nor $C_{x_2}$, the property is true for $N_1 = N_{20}$, $N_2 = N_{10}$. Finally, suppose that $(i, j)$ is a permutation of $(1, 2)$ such that $N_{10}$ contains $C_{x_i}$ and $N_{20}$ contains $C_{x_j}$. Since $N_{10}$ contains components of $M - K$ distinct from $C_{x_i}$, the closed point set $N_{10}$ is not connected. Therefore, $N_{10} = N_{101} + N_{102}$, where $N_{101}$ and $N_{102}$ are mutually exclusive closed point sets and $N_{101}$ contains $C_{x_j}$. The above stated property is then true for $N_1 = N_{101} + N_{20}$, $N_2 = N_{102}$. Now since $N_1$ contains both $C_{x_1}$ and $C_{x_2}$, we see that $N_2$ is a subset of $M_0$ and hence contains no limit point of $M - N$. Consequently, $N_2$ and $M - N_2$ are mutually separated, which is a contradiction to the assumption that $M$ is a continuum.

On the assumption that conclusion $(\beta)$ of Lemma 1 is false we have thus been led to a contradiction. Hence this conclusion is true, and Lemma 1 is established.

4. Remarks. In a letter to me, J. H. Roberts has stated that he has discovered an example of a continuum in three-dimensional euclidean space showing that the result of the theorem of this paper does not hold when the condition that the continuum $M$ be a plane continuum is omitted. There still remains the interesting question as to whether or not the statement obtained by replacing in our theorem the word "component" by "maximal strongly connected subset"* is true. That this latter question may also be answered in the negative when the condition that the continuum $M$ be a plane continuum is omitted is a consequence of an example given by Knaster and Kuratowski of a three-dimensional indecomposable continuum each of whose composants is a continuum.†

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* A point set $N$ is said to be strongly connected if for every two points $x$ and $y$ of $N$ there exists a continuum which contains both $x$ and $y$ and which is a subset of $N$. A maximal strongly connected subset of $N$ is a strongly connected subset of $N$ which is not a proper subset of any other strongly connected subset of $N$.