DISTRIBUTION OF MASS FOR AVERAGES OF NEWTONIAN POTENTIAL FUNCTIONS

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1. Introduction. It has been proved that the average of a potential function over a spherical volume and the average of a potential function over a spherical surface are themselves potential functions.* This paper is concerned with the determination of the distribution of mass for these two spherical averages; in addition, the distribution of mass for more general averages is obtained.

2. Preliminary Theorems. The problem is solved by means of a theorem on the change of the order of integration of an iterated Stieltjes integral. First it is necessary to state some preliminary theorems. We recall the following elementary theorem.

If \( h(Q) \) is continuous in \( Q \) and \( g(e) \) is a distribution of positive mass, bounded in total amount, and lying on a bounded set \( F \) (which may be taken as closed without loss of generality), then, for the integral over the whole of space, \( w \),

\[
\left| \int_{w} h(Q) \, dg(Q) - \sum_{\omega} h(Q_i) g(e_i) \right| < \omega_1 \alpha,
\]

where the summation is extended over all the meshes of a lattice \( L_\delta \), of diameter \( \leq \delta \), \( Q_i \) is a point of the mesh \( e_i \), \( \omega_1 \) is the oscillation of \( h(Q) \) on a subset of \( F \) of diameter \( \leq \delta \), and \( \alpha \geq g(F) \).

This theorem will be applied to the integral

\[
\int_{w} h^N(M, Q) \, dg(Q, P),
\]

where \( h^N(M, Q) \) is continuous in \( M, Q \), and \( g(e, P) \) and \( F \) are bounded independently of \( P \), so that \( \omega_1 \) and \( \alpha \) in (1) are independent of \( M, P \).

**Theorem 1.** If \( g(e, P) \) is a distribution of positive mass, bounded independently of \( P \), on a set \( F \) bounded independently

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of $P$, and if $f(e)$ is a distribution of positive mass, bounded in all space, and if $g(e, P)$ is summable with respect to $f(e)$, then

$$G(e) = \int g(e, P)df(e_P)$$

is a distribution of positive mass bounded in total amount and lying on the set $F$.

In this theorem, it is not required that $g(e, P)$ be continuous in $P$, so that the integral must be interpreted as a generalized or Daniell integral with respect to $f(e)$. But in the theorems which follow, $g(e, P)$ is taken as continuous in $P$ for every mesh of $L$.

To prove that $G(e)$ is a distribution of positive mass, we must show that $G(e)$ is a non-negative, absolutely additive function of point sets (only Borel measurable point sets are considered).

The first requirement follows immediately from the fact that $g(e, P)$ and $f(e)$ are non-negative for all $P$ and $e$. In order to exhibit the second condition, let $e = e_1 + e_2 + \cdots$ and $e'_n = e_1 + e_2 + \cdots + e_n$; as $g(e, P)$ is a distribution of positive mass, we have

$$g(e, P) = \lim_{n \to \infty} g(e'_n, P), \text{ with } g(e'_{n+1}, P) \geq g(e'_n, P).$$

Since $g(e, P)$ is, as a function of $P$, the limit of an increasing sequence of functions, the order of integration and passing to the limit may be interchanged; so that

$$G(e) = \int g(e, P)df(e_P) = \lim_{n \to \infty} \int g(e'_n, P)df(e_P)$$

$$= \lim_{n \to \infty} [G(e_1) + \cdots + G(e_n)] = G(e_1) + G(e_2) + \cdots.$$ 

Hence $G(e)$ is a distribution of positive mass. Also $G(e)$ is a distribution of mass lying on $F$; that is, $G(e) = 0$, if $e \cdot F = 0$, because of the hypothesis that $g(e, P) = 0$, if $e \cdot F = 0$. Finally, $G(e) \leq \text{[upper bound of } g(e, P) \cdot f(w)\text{], so bounded.}$

We state the generalizations of two theorems proved by H. E. Bray; the proofs are omitted as they are essentially the same as those given by Bray.* They depend on the inequality (1).

THEOREM 2. If $h^N(M, Q)$ is a continuous function of $M$ and $Q$, and is bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of $P$, on a set $F$ bounded independently of $P$, and if $g(e, P)$ is continuous in $P$ for every cell $e$ of a net $L$, then

$$K(M, P) = \int_{\omega} h^N(M, Q) dg(e_Q, P)$$

is continuous in $M$ and $P$.

THEOREM 3. If $h^N(M, Q)$ is continuous in $M$ and $Q$, and bounded in all space, if $g(e, P)$ is a distribution of positive mass, bounded independently of $P$, on a set $F$ bounded independently of $P$, and if it is continuous in $P$ for every cell $e$ of a net $L$, and if $f(e)$ is a distribution of positive mass, bounded in total amount, lying on a bounded set $E$ (which may be taken as closed without loss of generality), then the integrals

$$\int_{\omega} df(e_P) \int_{\omega} h^N(M, Q) dg(e_Q, P),$$

and

$$\int_{\omega} h^N(M, Q) d \left[ \int_{\omega} g(e_Q, P) df(e_P) \right]$$

exist and are equal.

We may now state and prove the concluding theorem in this series.

THEOREM 4. If $g(e, P)$ is a positive distribution of mass, bounded in total amount independently of $P$, on a set $F$ bounded independently of $P$, and if it is continuous in $P$ for every cell $e$ of the net $L$, and if $f(e)$ is a distribution of positive mass, bounded in total amount and lying on the set $E$, then

$$\int_{\omega} df(e_P) \int_{\omega} \frac{1}{M^Q} dg(e_Q, P) = \int_{\omega} \frac{1}{M^Q} d \left[ \int_{\omega} g(e_Q, P) df(e_P) \right],$$

(or both are $+\infty$), where $\int_{\omega} g(e, P) df(e_P)$ is a bounded distribution of positive mass lying on the set $F$. 
If

\[ h^N(M, Q) = \frac{1}{MQ}, \quad \text{if } \frac{1}{MQ} \leq N, \]

\[ = N, \quad \text{if } \frac{1}{MQ} > N, \]

the function \( h^N(M, Q) \) is continuous in \( M \) and \( Q \).

By Theorem 3, we have

\[
\int \omega df(e_P) \int \omega h^N(M, Q) dg(e_Q, P) = \int \omega h^N(M, Q) d \left[ \int \omega g(e_Q, P) df(e_P) \right].
\]

Let \( N \) become infinite; we have

\[
\int \omega df(e_P) \lim_{N \to \infty} \int \omega h^N(M, Q) dg(e_Q, P) = \int \omega \frac{1}{MQ} d \left[ \int \omega g(e_Q, P) df(e_P) \right].
\]

The interchange of integration and passing to the limit in the left-hand member is justified because the integrand is an increasing function of \( N \), while the definition of the integral of a lower-semi-continuous function with respect to a distribution of positive mass was used in the right-hand member. Applying this definition now to the left-hand member, we have

\[
\int \omega df(e_P) \int \omega \frac{1}{MQ} dg(e_Q, P) = \int \omega \frac{1}{MQ} d \left[ \int \omega g(e_Q, P) df(e_P) \right].
\]

This, combined with Theorem 1, establishes the theorem.

3. Volume Averages. This theorem just proved enables one to exhibit the distribution of mass for the spherical volume average of a potential function in a form in which it may be evaluated; this is an illustration of the advantage sometimes gained by working with the more general situation.

In the theorems that follow, \( u(Q) \) is the potential at \( Q \) of the distribution of positive mass, \( f(e) \), which is bounded in total amount and lies on a bounded set \( E \).
Theorem 5. The average of the potential function $u(Q)$ over a spherical volume $\Gamma(r, M)$, of radius $r$ and center $M$, is a potential function of a distribution of positive mass with density $(3/(4\pi r^2))f\{\Gamma(r, Q)\}$. In symbols,

$$a_u(r, M) = \frac{3}{4\pi r^3} \int_{\Gamma(r, M)} u(Q)dQ$$

(2)

Let us form this spherical volume average and determine its distribution of mass. We have

$$a_u(r, M) = \frac{3}{4\pi r^3} \int_{\Gamma(r, M)} u(Q)dQ$$

$$= \frac{3}{4\pi r^3} \int_{w} \frac{1}{MQ} f\{\Gamma(r, Q)\}dQ.$$ 

As the integrand is a lower-semi-continuous function, the order of integration may be interchanged, so that

$$a_u(r, M) = \int_{w} \frac{1}{PQ} df(e_P).$$

The inner integral is the potential at $P$ of a sphere of unit density with radius $r$ and center at $M$, which is equal to the potential at $M$ of the same sphere with center at $P$. Hence,

$$a_u(r, M) = \frac{3}{4\pi r^3} \int_{w} df(e_P) \int_{\Gamma(r, P)} \frac{1}{PQ} dQ.$$

where $g(e, P) = m_3\{e \cdot \Gamma(r, P)\}$, and $m_3$ means the three-dimensional measure of the set indicated. The quantity $g(e, P)$ is evidently continuous in $P$ for every measurable set $e$.

By means of Theorem 4, this volume average may be expressed in a form which enables the mass distribution to be evaluated,
Consider the integral,
\[ v(\epsilon) = \int_{\omega} m_3\{ \epsilon \cdot \Gamma(r, P) \} d\epsilon, \]
where \( \epsilon \) is an arbitrary bounded set measurable Borel. This function is an absolutely continuous function of \( \epsilon \); in fact
\[ v(\epsilon) \leq m_3(\epsilon) \cdot f(E), \]
and \( v(\epsilon) \) is completely additive by Theorem 1.

The integrand of \( v(\epsilon) \) may be expressed as a Lebesgue integral,
\[ m_3\{ \epsilon \cdot \Gamma(r, P) \} = \int_{\epsilon \cdot \Gamma(r, P)} 1dR = \int_{\epsilon} B(R, P)dR, \]
where we may define \( B(R, P) \) as 1, for \( RP < r \), and 0, for \( RP \geq r \). In this way \( B(R, P) \) is the limit of an increasing sequence of continuous functions of \( R \) and \( P \), and we have
\[ v(\epsilon) = \int_{\omega} df(\epsilon, P) \int_{\epsilon} B(R, P)dR = \int_{\epsilon} dR \int_{\omega} B(R, P)df(\epsilon, P) \]
(4)
\[ = \int_{\epsilon} f(\Gamma(r, R))dR. \]

Substituting this result in (3) and making use of the definition of the integral of a lower-semi-continuous function, we have
\[ a_u(r, M) = \frac{3}{4\pi r^3} \int_{\omega} \frac{1}{MQ} \left[ \int_{\epsilon} f(\Gamma(r, R))dR \right] \]
\[ = \frac{3}{4\pi r^3} \lim_{N \to \infty} \int_{\omega} h^N(M, Q)d \left[ \int_{\epsilon} f(\Gamma(r, R))dR \right]. \]

As \( h^N(M, Q) \) is bounded and continuous, and \( v(\epsilon) \) is absolutely continuous, we may change the Stieltjes integral to a Lebesgue integral. This gives
\[ a_u(r, M) = \frac{3}{4\pi r^3} \lim_{N \to \infty} \int_{\omega} h^N(M, Q)f(\Gamma(r, Q))dQ \]
\[ = \frac{3}{4\pi^3} \int_{\mathcal{C}} \frac{1}{MQ} f\{\Gamma(r, Q)\} dQ, \]

and the proof is complete.

It should be pointed out that the work of this section holds for the average over any three-dimensional open set. Let \( s(O) \) be an open set (therefore of positive spatial measure), \( s(M) \) the set obtained by displacement of \( s(O) \) as a rigid body, without rotation, so that \( O \) falls on \( M \), and let \( s'(Q) \) be the reflection of the set \( s(M) \) through the midpoint of the line \( MQ \).

**Theorem 6.** The average of the potential function \( u(Q) \) over the set \( s(M) \) is a potential function of a distribution of positive mass with density \( \left( \frac{1}{m_3(s(M))} \right) f\{s'(Q)\} \). In symbols,

\[ a_u\{s(M)\} = \frac{1}{m_3\{s(M)\}} \int_{s(M)} u(Q) dQ = \frac{1}{m_3\{s(M)\}} \int_{s(M)} \frac{1}{MQ} f\{s'(Q)\} dQ. \]

The construction of the set \( s'(P) \) gives the potential at \( P \) of the set \( s(M) \), of unit density, equal to the potential at \( M \) of the set \( s'(P) \), of unit density; hence all the transformations made in this section on the spherical volume average are valid for the average over the set \( s(M) \).

4. Spherical Surface Average. By means of Theorem 4, we may also determine the mass function for the average of a potential function over a spherical surface.

**Theorem 7.** The average of the potential function \( u(Q) \) over the spherical surface \( C(r, M) \), of radius \( r \) and center \( M \), is a potential function of the distribution of positive mass,

\[ \frac{1}{4\pi r^2} \int_{C(r, M)} m_2\{e \cdot C(r, P)\} df(e_P). \]

In symbols,

\[ A_u(r, M) = \frac{1}{4\pi r^2} \int_{C(r, M)} u(Q) dQ = \frac{1}{4\pi r^2} \int_{\mathcal{C}} \frac{1}{MQ} d \left[ \int_{\mathcal{C}} m_2\{e_Q \cdot C(r, P)\} df(e_P) \right]. \]
Treating the spherical surface average in the same manner as
we have treated the volume average, we obtain
\[
A_u(r, M) = \frac{1}{4\pi r^2} \int_{C(r,M)} u(Q)dQ = \frac{1}{4\pi r^2} \int_{C(r,M)} dQ \int_0^1 \frac{1}{QP} df(e_P)
\]
\[
= \frac{1}{4\pi r^2} \int_0^1 df(e_P) \int_{C(r,M)} \frac{1}{QP} dQ
= \frac{1}{4\pi r^2} \int_0^1 df(e_P) \int_{C(r,P)} \frac{1}{QM} dQ
= \frac{1}{4\pi r^2} \int_0^1 df(e_P) \int_0^1 \frac{1}{MQ} dg(e_Q, P),
\]
where \(g(e, P) = m_2 \{ e \cdot C(r, P) \} \). For a given \(P\), this function is
additive and bounded for cells \(e\) of a three-dimensional lattice,
and hence can be extended by definition uniquely to all sets
spatially measurable Borel.

As \(g(e, P)\) is a continuous function of \(P\) for every cell \(e\) of a
net \(L\), Theorem 4 applies, and we may thereby express this aver­
age in a form that exhibits its mass function in terms of \(C(r, P)\)
and \(f(e)\),
\[
(5) \quad A_u(r, M) = \frac{1}{4\pi r^2} \int_0^1 \frac{1}{MQ} d \left[ \int_0^1 m_2 \{ e_Q \cdot C(r, P) \} df(e_P) \right].
\]
This result requires no restriction on \(f(e)\) other than those we
have already stated. However, we shall state also a special case
of Theorem 7.

**Theorem 8.** If \(f(\Gamma(r, Q))\) is derivative with respect to \(r\), and,
for a fixed neighborhood of the given value of \(r\), \(\partial f(\Gamma(n, Q))/\partial r\)
is bounded independently of the point \(Q\), then \(\text{the average of the}
potential function } u(Q) \text{ over the spherical surface } C(r, M) \text{ is a}
potential function of a distribution of positive mass with density
\((1/(4\pi r^2))\partial f(\Gamma(r, Q))/\partial r\). In symbols,
\[
(6) \quad A_u(r, M) = \frac{1}{4\pi r^2} \int_0^1 \frac{1}{MQ} \frac{\partial f(\Gamma(r, Q))}{\partial r} dQ.
\]
We have, for a rectangular cell \(e\),
\[
(7) \quad m_2 \{ e \cdot C(r, P) \} = \lim_{i=\infty} \frac{m_3 \{ e \cdot [\Gamma(r_i, P) - \Gamma(r_i, P)] \}}{r_i - r},
\]
where \(r < r_{i+1} < r_i\) and \(\lim_{i=\infty} r_i = r\).
Using the results given in (4), we have the following equality,

\[ \int m_w \{ e \cdot [\Gamma(r_i, P) - \Gamma(r, P)] \} \frac{df(e_P)}{r_i - r} = \int f\{\Gamma(r_i, Q)\} - f\{\Gamma(r, Q)\} \frac{dQ}{r_i - r}. \]

The integrand of the left-hand member belongs to a sequence of measurable, uniformly bounded functions, as a function of \( P \), whose limit exists when \( i \) becomes infinite; so we let \( i \) become infinite and interchange the order of integration and pass to the limit for the left-hand member. The same considerations hold for the integrand of the right-hand member as a function of \( Q \). Using (13), we have

\[ \int m_w \{ e \cdot C(r, P) \} df(e_P) = \int e \frac{\partial f\{\Gamma(r, Q)\}}{\partial r} dQ. \]

The quantity \( e \frac{\partial f\{\Gamma(r, Q)\}}{\partial r} \) is non-negative. Hence we may substitute this last equation in (5) and change the Stieltjes integral into a Lebesgue integral as we did above for the volume average. Thus we have established the theorem.

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**ERRATUM**

In my paper entitled *On the summability of a certain class of series of Jacobi polynomials* (this Bulletin, vol. 41 (1935), pp. 541–549), the following change should be made; it conforms with the last proofs that I had seen.

Page 544, 8th line from the bottom: read \( S_{n,k}^{(k)} \) instead of \( S_{n,k}^{(k)} \).

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