

BAKER ON GEOMETRY

Principles of Geometry. By H. F. Baker. Cambridge University Press. Volumes 1–6, 1922–1933.

Volumes 2 and 3 of this series have been reviewed separately, but there has been no review in this Bulletin of Volumes 1 and 4. Volumes 1 and 2 have been recently reprinted with slight revisions. The last two volumes were published in the latter part of 1933.

A statement by the author in the preface of Volume 5 may be taken to indicate that the series ends with Volume 6. Also, upon inquiry, the publishers have written that, so far as they know, second editions of Volumes 3 and 4 are not in preparation and the set is complete in six volumes. Mathematicians, however, will rejoice if these inferences are in error and additional volumes eventually appear. Two years ago, one might have inferred equally well that the series had ended with Volume 4!

The volumes are all neatly and durably bound in dark blue cloth. The printing and paper are excellent. The pages, however, seem over-crowded. They would be more pleasing to the reader if there were a larger number of paragraphs, more displayed equations, and if the examples, especially the longer ones in the later volumes, were in larger type.

All the volumes are indexed. In the later volumes, the indices are very good, but in the first volume, the index is far too brief to be of much use. The section titles within a chapter are pithily descriptive of the subject matter in the associated section. These section titles, with page references, are printed in full in the contents. For this reason, the contents are often of as much or more value than the index in looking up a subject reference.

The advertising leaflet issued by the publishers calls the set "a mine of geometrical lore." This describes it well. It is, indeed, a veritable gold mine, but no mine yields its treasure for nothing—one must work hard to gain it. Likewise, he who attempts merely to read these books will get little or nothing, but he who is willing to dig and delve in this mine, to amplify in scope, and complete in detail, will be repaid many times over for his time and effort by the real, but none too accessible, treasure that lies herein.

We have no quarrel with the author for maintaining the tradition that "there is no royal road to geometry." For, although the origin of this statement is traditional, the fact expressed is true of all mathematics, and all else worth while. There are certain unnecessary difficulties, however, due chiefly to arrangement. When new terms or symbols are introduced, they are often defined in obscure places to which there are usually no references in the index. The reader must scan many pages carefully to find the definitions of the symbols or terms whose meanings he can not conjecture from the context. Also, a new term is sometimes used before it is defined.

For example, in Volume 5, page 109, line 8 from top, we find, "A cubic primal in space of four dimensions contains thirty-five terms." The word "primal" is used occasionally from now on and is finally defined (p. 182),

parenthetically, in a footnote with nothing to indicate in the paragraph or sentence to which the footnote is appended that mention would be made of this term or of the idea it represents.

The new terms "prime" (Vol. 4, p. 219) and "primal" (Vol. 5, p. 182, footnote) for what have been called, none too happily, "hyperplane" and "hyper-surface," respectively, evidently occurred to the author after a part of the set had been written. Both of these are timely and excellent contributions to mathematical terminology. The names "hyperplane" and "hypersurface" are unwieldy, inappropriate, and misleading—why not, as logically, "hyperline" or "hyperpoint" and "hypercurve" to designate these ideas!

Volume 1 contains a valuable bibliography. It is regrettable that this excellent feature occurs only in a modified form in Volume 2 and is entirely discontinued in succeeding volumes. In Volumes 5 and 6, however, many references are given at the end of sections and chapters.

The individual volumes will now be discussed.

Volume 1. *Foundations*. First edition, 1922. xi+183 pp. Second edition, 1929. xi+195 pp.

This volume presents the fundamentals of projective geometry, primarily in the plane and in three-space, with extensions to any number of dimensions.

Synthetic methods are used chiefly in this and in most of the succeeding volumes. The author's point of view as to method is well expressed in the following excerpt from the preface: "The geometrical theory is accompanied by an algebraic symbolism, which serves to help to fix ideas, and for purposes of verification; it is necessary also to include the proof that this symbolism is appropriate to the purpose. It is held, however, that the geometrical argument should be complete in itself, independently of the symbols; a geometry should have such a comprehensive grasp of geometrical relations that all its results are clear by consideration of the geometrical entities alone."

The following changes occur in the second edition:

The corrections given in Volume 2, first edition, page 238, and in Volume 3, page 224, are made in the text.

Three notes, on pages 184–194, are added, amplifying certain stated portions of this volume. In each of the lists at the end of chapters 2 and 3 occurs one additional problem. The only erratum (p. viii) refers to the second of these problems.

The excellent bibliography, on pages 176–183, is revised and slightly enlarged.

Volume 2. *Plane Geometry. Conics, Circles, Non-Euclidean Geometry*. First edition, 1922. xv+243 pp.* Second edition, 1930, xv+259 pp.

The following changes occur in the second edition:

The corrections given on page 224, Volume 3 are made in the text.

The bibliography of Notes 2 and 3 is slightly enlarged.

More extended discussions (Additions, pp. 239–253) of certain problems and theorems of the text account for the additional pages of the second edition. In the body of the book, these are referred to by the abbreviation "Add." appended to the problem or theorem to be further considered.

* Reviewed by F. S. Woods, this Bulletin, vol. 31 (1925), pp. 370–371.

Volume 3. *Solid Geometry. Quadrics, Cubic Curves in Space, Cubic Surfaces.* 1923. xix+228 pp.*

Corrections to this volume are given in Volume 4, p. 244.

Volume 4. *Higher Geometry. Being illustrations of the utility of the consideration of higher space, especially of four and five dimensions.* 1925. xvi+250 pp.

The author states in the preface that the fourth volume was written first and that the other three are chiefly introductory to it.

In this volume, the last three chapters are the most important. Chapters 1 and 3 are introductory to these. Chapters 2 and 4, however, are inserted because of their own interest and value and are not necessary for succeeding chapters. For this reason, these two chapters will be briefly treated before taking up the main sequence.

Chapter 2 is devoted to the discussion of the Hart circle, a circle determined by any three given circles such that four other circles touch this circle and each of the original three. All eight circles are in one plane.

Chapter 4 deals with the configuration in S_4 corresponding to the set of twenty-seven lines on the cubic surface in S_3 .

In Chapter 1, certain basic relations are treated, first in two and three, then in three and four, and finally in five dimensions. At first glance, the arrangement of topics seems haphazard, but evidently they have been very carefully chosen by the author in carrying out his purpose to give, not a complete discussion, but a sufficient number of selected illustrations to enable the interested reader to fill in and complete the picture himself.

Chapters 3 and 6 are so closely related that they are best treated together. In Chapter 3 are developed the properties of the elliptic plane quartic, obtained by projection of the intersection quartic of two quadrics of S_3 . This is done in considerable detail to serve as a basis for Chapter 6 in which the cyclide, a special quartic surface of S_3 , is studied as the projection of the intersection surface of two quadric primals of S_4 . Dupin's cyclide and other special forms are included in the discussion.

Chapter 5 takes up the study of the configuration of fifteen lines and fifteen points in S_4 (see frontispiece) and the associated loci, many of which are old friends introduced in a new way.

Chapter 7 deals chiefly with relations in five dimensions associated with Kummer's quartic surface.

The treatment throughout is largely synthetic. In general, topics are developed by consecutive examples, which are stated and either are solved or have directions given for their solutions. Often the connecting discussion between adjacent examples must be supplied by the reader.

This volume covers a wide range of material, chosen with a purpose and accomplishing that purpose—to demonstrate the well nigh indispensable utility of higher space in the derivation and study of properties of configurations in ordinary space.

Volume 5. *Analytical Principles of the Theory of Curves.* 1933. x+247 pp.

The last two volumes, which appeared almost simultaneously, are extremely rich in variety and scope. In the preface of Volume 5, the author tells why these two volumes were written.

* Reviewed by B. H. Brown, this Bulletin, vol. 32 (1926), pp. 173-174.

"In the last fifty years, a remarkable advance has been made in the theory of surfaces, and of algebraic loci in general; the English reader may find a description of the nature of this in a Presidential Address to the London Mathematical Society given* in November, 1912. But attempts, since the War, to expound these new results have continually shown the necessity for a precise appreciation of the ideas out of which this advance has developed; in mathematics it is not sufficient to know the enunciation of a result; it is necessary to understand the proof. These two volumes have grown up in the attempt to meet this need."

Volumes 5 and 6 differ from the preceding volumes in that algebra is freely used and the discussion follows the more usual form with examples employed chiefly for illustration or extension.

The author assumes that the reader has a knowledge of the projective properties of higher plane curves. There has been no adequate treatment, in fact, scarcely any treatment of this subject, not even of plane cubics and quartics (except the elliptic quartic) in the preceding volumes. This, however, is in line with the author's statement that he does not attempt to complete any subject and with his evident aim not only to impart information, but to stimulate the reader to do as much of the necessary work himself as possible.

In the first chapter, birationally equivalent manifolds are defined. Rational and elliptic plane curves are discussed as the two simplest examples of such equivalence. In Chapter 2, the linear series on a curve, defined in Chapter 1, is used to reduce a curve with given singularities to a non-singular curve.

For the next five chapters, some familiarity with the theory of functions is essential. The algebraic functions for a curve and Abel's Theorem for the sum of algebraic integrals are discussed. The integral genus is defined and is shown to be identical with the arithmetical genus. The linear series on a curve is now treated in more detail with applications and examples in the four notes (pp. 86-110). Chapters 5 and 6 deal with algebraic integrals, especially in regard to their periods and the relation of these to Riemann surfaces.

In Chapter 7, the modular forms of rational functions and integrals are introduced and used to derive anew some of the results of the preceding chapters.

In the first part of the final chapter, the relations among the properties of curves of S_3 are derived by projective methods and certain relations are shown to have interpretations in the light of Chapter 4. The relations for space curves which are complete or partial intersections of two surfaces are now obtained and the linear series on a space curve studied. The chapter closes with a few applications to curves of higher space.

Volume 6. *Introduction to the Theory of Algebraic Surfaces and Higher Loci*. 1933. ix+308 pp.

In this volume, the first three chapters deal with the theory of correspondence and the last four with surfaces in three and four dimensions.

In Chapter 1, point correspondences on the same curve or on two curves are the chief topics of the first part; part 2 introduces transcendental methods

* Proceedings of the London Mathematical Society, vol. 12 (1913), pp. 1-40.

and part 3, defective integrals, with applications to the theory of correspondence.

In the second chapter, the theory of correspondence is extended to points and primes in higher space. The treatment employs the method and notation of Schubert's symbolic calculus.

Chapter 3 deals briefly with transformations and involutions of the plane, including a short resume of Bertini's four types. Many references to the extensive literature on this subject are given in the course of the discussion.

In Chapter 4, relations of projective properties of surfaces in three and four dimensions are treated, with numerous examples and references. On page 157, the author introduces a new term, "accidental double points," for the term, "improper double points," used by Severi in the paper cited in the footnote on this page. An excellent reason is given (p. 158) for Severi's choice of a name for these points, but no grounds are offered for the rejection of this name and the substitution of another. Moreover, the use of the adjective, "accidental," in this connection seems inappropriate, since all surfaces of S_4 which are projections of surfaces of S_6 (with the exception of cones and the Veronese quartic) have improper double points, that is, such a point on a surface in S_4 is no more accidental than the double curve of a surface in S_3 or the double point of a plane curve.

Chapter 5 deals with the theory of invariants of birational transformations of a surface in three dimensions. In the preface, the author states that this chapter "is concerned with the most interesting and the most novel ideas of the volume." The discussion is intended to be only introductory and many references are given for those who wish to continue it further.

In Chapter 6, the intersections of loci in S_4 , especially primals, are treated in some detail.

The last chapter contains particular theorems and examples, and also applications of some of the theory developed in the preceding chapters.

This series is the most comprehensive treatise on geometry that has ever appeared in English. To be sure, there are many gaps, such as the inadequate treatment of higher plane curves, but these omissions consist chiefly of routine matter, amply covered in several current treatises in English and other languages. Volumes 4, 5, and 6 contain the principal contributions of the series.

This work is also remarkable because of its emphasis on the synthetic method. The attitude of the author that geometry should, of itself, be able to justify its conclusions reminds one of Steiner. The author, however, evidently does not "hate" analysis, as Steiner is said to have done, but uses it with energy and finesse whenever he deems it preferable.

Mathematicians, especially geometers, are deeply indebted to Professor Baker for his invigorating presentation of this basic branch of our science.

T. R. HOLLCROFT