SOME MULTIPLICATION THEOREMS
FOR THE NÖRLUND MEAN

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Absolute summability for the series $\sum_{n=1}^{\infty} u_n$ by the Cesàro mean and by the Riesz mean have been defined by Fekete* and by Obrechkoff,† respectively. In each case, theorems for the multiplication of series summed by these means have been proved.‡ The purpose of this paper is to establish a definition for absolute summability by the Nörlund mean, and to prove three multiplication theorems for this mean. Theorem 1 includes Mertens' theorem for convergent series and its extension for the Cesàro mean. Theorem 2 includes Cesàro's multiplication theorem. Theorem 3 includes the following theorem by M. J. Belinfante for the Cesàro mean.

If $\sum_{n=1}^{\infty} u_n$ is summable $C_s$ to $U$, and if $\sum_{n=1}^{\infty} v_n$ is summable $C_r$ to $V$, and bounded $C_{r-1}$, $(s \geq 0, r \geq 1)$, the product series $\sum_{n=1}^{\infty} w_n$ is summable $C_{r+s}$ to $UV$.§

For any given series $\sum_{k=1}^{\infty} u_k$, with terms real or complex, form the sequence $\{ U_k \}$, where $U_k = \sum_{n=1}^{k} u_n$. Let $\{ a_n \}$ be a sequence of positive numbers, and let $A_n = \sum_{k=1}^{n} a_n$. The series $\sum_{k=1}^{\infty} u_k$ is said to be summable to $U'$ by the Nörlund mean $A$ if

$$\lim_{n \to \infty} U_n' = \lim_{n \to \infty} \frac{1}{A_n} \sum_{k=1}^{n} a_{n-k+1} U_k$$

exists and is equal to $U'$.¶ If $\sum_{k=1}^{\infty} u_k'$, where $u_k' = U_k' - U_{k-1}'$, is absolutely convergent, we shall say that $\sum_{k=1}^{\infty} u_k$ is absolutely summable $A$. We shall assume that $\lim_{n \to \infty} (a_n/A_n) = 0$; then $A$ is a regular method of summation.||

‡ For discussion and references, see Kogbetliantz, Mémorial des Sciences Mathématiques, No. 51.
§ Koninklijke Akademie te Amsterdam, Verslag, vol. 32 (1923), pp. 177–189.
|| Riesz, loc. cit.
We shall consider also the series $\sum_{k=1}^{\infty} v_k$, and $\sum_{k=1}^{\infty} w_k$, the Cauchy product of series $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$; we have the corresponding sequences $\{V_k\}$ and $\{W_k\}$.

We shall assume that we have also a regular Nörlund mean, $B$, defined by $\{b_k\}$, a sequence of positive numbers. We shall form the Nörlund means, $C$, defined by $\{c_n\} = \{\sum_{k=1}^{n} a_k b_{n-k+1}\}$, and $D$, defined by $\{d_n\} = \{\sum_{k=1}^{n} a_k b_{n-k+1}\}$.

**Theorem 1.** If $\sum_{k=1}^{\infty} u_k$ is summable $A$ to $U'$, and in addition, absolutely summable $A$ and if $\sum_{k=1}^{\infty} v_k$ is summable $B$ to $V'$, then $\sum_{k=1}^{\infty} w_k$ is summable $C$ to $U' V'$.

**Proof.** We shall prove the theorem first with the assumption that $V' = 0$. Let

$$
U'_n = \frac{1}{A_n} \sum_{k=1}^{n} a_k U_{n-k+1}, \quad V'_n = \frac{1}{B_n} \sum_{k=1}^{n} b_k V_{n-k+1},
$$

$$W'_n = \frac{1}{C_n} \sum_{k=1}^{n} c_k W_{n-k+1};$$

let $\sum_{k=1}^{\infty} u'_k$, $\sum_{k=1}^{\infty} v'_k$, and $\sum_{k=1}^{\infty} w'_k$ be the corresponding series; then

$$|W'_n| \leq \frac{1}{C_n} \left\{ \left| u'_1 \right| \left| \sum_{l=1}^{n} c_l V_{n-l+1} \right| + \sum_{k=2}^{n} \left| u'_k \right| \left( \sum_{l=1}^{n-k} V_l S + A_k b_1 V_{n-k+1} \right) \right\},$$

where

$$S = A_k b_{n-k-l+2} + \sum_{p=k+1}^{n-l+1} a_p b_{n-l-p+2}.$$

Hence

$$|W'_n| \leq \frac{1}{C_n} \left\{ \left| u'_1 \right| \left| \sum_{l=1}^{n} c_l V_{n-l+1} \right| \right.$$

$$\left. + \sum_{k=2}^{n} \left| u'_k \right| \left| A_k \right| \left( \sum_{l=1}^{n-k+1} b_{n-k-l+2} V_l \right) \right.$$  

$$\left. + \sum_{k=2}^{n-1} \left| u'_k \right| \left( \sum_{l=1}^{n-k} \sum_{p=k+1}^{n-l+1} a_p b_{n-l-p+2} V_l \right) \right\}$$

$$= P_n \left| u'_1 \right| + \sum_{k=2}^{n} \left[ \left| u'_k \right| Q_{nk} \right] + \sum_{k=2}^{n-1} \left[ \left| u'_k \right| R_{nk} \right].$$
Since $C$ includes $B,*$

$$\lim_{n \to \infty} P_n = 0.$$  

For $2 \leq k \leq n$,

$$Q_{nk} < \frac{\sum_{l=k}^{n} A_l b_{n-l+1}}{\sum_{l=k}^{n} A_l b_{n-l+1}} = \left| V'_{n-k+1} \right|.$$  

Therefore

$$\sum_{k=2}^{n} Q_{nk} \left| u_k' \right| < \sum_{k=2}^{n} \left| V'_{n-k+1} \right| \left| u_k' \right|$$

$$= \sum_{k=2}^{n} \left| V'_{n-k+1} \right| \left| u_k' \right| + \sum_{k=n+1}^{n} \left| V'_{n-k+1} \right| \left| u_k' \right|,$$

where $\nu$ may be chosen so that $\nu$ and $n - \nu$ become infinite with $n$. Since $\lim_{n \to \infty} \left| V'_n \right| = 0$, for any $\epsilon, \nu$ and $n$ may be chosen sufficiently large so that

$$\sum_{k=2}^{n} \left[ Q_{nk} \left| u_k' \right| \right] < \epsilon.$$

For $2 \leq k \leq n-1$, we have

$$R_{nk} \leq \frac{\sum_{l=1}^{n-2} a_{n-l+1} B_l \left| V'_l \right|}{C_n}$$

$$= \left[ \frac{1}{C_n} \sum_{l=1}^{n-2} a_{n-l+1} B_l \right] \left[ \sum_{p=1}^{n-2} \frac{a_{n-p+1} B_p}{\sum_{l=1}^{n-2} a_{n-l+1} B_l} \left| V'_p \right| \right]$$

$$< \sum_{p=1}^{n-2} \frac{a_{n-p+1} B_p}{\sum_{l=1}^{n-2} a_{n-l+1} B_l} = V''_{n-2},$$

* Nörlund, Lunds Universitet, Årsskrift, (2), vol. 16 (1919), No. 3.
where \( V'_{n-2} \) is the \((n-2)\)th term of the sequence obtained when \( \{ |V'_p| \} \) is summed by the matrix transformation \( t_{np} \), where

\[
t_{np} = \frac{a_{n-p+2}B_p}{\sum_{l=1}^{n} a_{n-l+3}B_l}
\]

for \( p \leq n \), and \( t_{np} = 0 \) for \( p > n \).

This transformation is regular, since \( A \) is regular; it follows that \( \lim_{n \to \infty} V''_{n-2} = 0 \), and that for any \( \epsilon \), we may find \( n \) sufficiently large so that

\[
R_n < \epsilon.
\]

From (1), (2), (3), and the fact that \( \sum_{k=1}^{\infty} |u'_k| \) converges, it follows that \( \lim_{n \to \infty} |W'_n| = 0 \). This proves the theorem for \( V' \).

If \( V' \neq 0 \), we consider the sequence \( \{ V_n - V' \} \); this sequence is summed by \( B \) to 0. Hence the Cauchy product of \( \sum_{n=1}^{\infty} u_n \) by \( [v_1 - V'] + \sum_{n=2}^{\infty} v_n \) is summed by \( C \) to 0; that is,

\[
\lim_{n \to \infty} \left[ W'_n - V' \frac{\sum_{k=1}^{n} c_k U_{n-k+1}}{C_n} \right] = 0.
\]

Since \( C \) includes \( A \),

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} c_k U_{n-k+1}}{C_n} = U';
\]

therefore \( \lim_{n \to \infty} W'_n = U'V' \).

**Theorem 2.** If \( \sum_{k=1}^{\infty} u_k \) is summable \( A \) to \( U' \), and if \( \sum_{k=1}^{\infty} v_k \) is summable \( B \) to \( V' \), then \( \sum_{k=1}^{\infty} w_k \) is summable \( D \) to \( U'V' \).

**Proof.** We have

\[
W'_n = \frac{1}{D_n} \sum_{k=1}^{n} d_k W_{n-k+1} = \sum_{k=1}^{n} g_{nk} U'_k V'_{n-k+1},
\]

where \( g_{nk} = A_k B_{n-k+1}/D_n \) for \( k \leq n \), and \( g_{nk} = 0 \) for \( k > n \).
Since $A$ and $B$ are regular, this method of summation is regular and $\lim_{n \to \infty} g_{n,n-k+1} = 0$; it follows that

$$\lim_{n \to \infty} \sum_{k=1}^{n} g_{nk} U'_k V'_{n-k+1} = U'V',$$

which completes the proof.

For the proof of Theorem 3 we require the following lemma.

**Lemma.** If $\{X_n\}$ and $\{[\sum_{k=1}^{n} B_k y_{n-k+1}] / B_n\}$ converge to $X$ and $Y$, respectively, and if $\{\sum_{k=1}^{n} b_k y_{n-k+1} / b_n\}$ is bounded, then

$$\lim_{n \to \infty} \left( \sum_{k=1}^{n} a_{nk} X_k \frac{\sum_{l=1}^{n-k+1} b_l y_{n-k-l+2}}{b_{n-k+1}} \right) = XY,$$

provided that (a) $\lim_{n \to \infty} a_{nk} = 0$; (b) $\sum_{k=1}^{n} |a_{nk}| < M$ for all $n$, where $M$ is a positive constant; (c) $T'$ includes $T$, where $T'$ and $T$ are triangular matrix transformations defined by $t'_n = a_{n,n-k+1}$ and $t_{nk} = b_k / B_n$.

**Proof.** Let

$$\sum_{l=1}^{n-k+1} b_l y_{n-k-l+2}$$

and let $a_{nk} y'_{n-k+1} = c_{nk}$. Let $Z_n = \sum_{k=1}^{n} c_{nk} X_k$. From (a), (b), and (c), it follows that $\lim_{n \to \infty} c_{nk} = 0$, $\sum_{k=1}^{n} |c_{nk}| < M'$, where $M'$ is a positive constant, and $\lim_{n \to \infty} \sum_{k=1}^{n} c_{nk} = Y$. Choose $p$ such that for a given $\epsilon > 0$, $|X_k - X| < \epsilon / (2M')$ when $k > p$. For $k \leq p$, $|X_k - X| < L$. Then

$$Z_n - X \sum_{k=1}^{n} c_{nk} \leq \sum_{k=1}^{p} |c_{nk}| |X_k - X| + \sum_{k=p+1}^{n} |c_{nk}| |X_k - X|$$

$$\leq L \sum_{k=1}^{p} |c_{nk}| + \frac{\epsilon}{2M'} |c_{nk}|.$$

Choose $N > p$, and such that $|c_{nk}| < \epsilon / (2pL)$ for $n > N$. Then for $n > N$, $|Z_n - X \sum_{k=1}^{n} c_{nk}| < \epsilon$, which proves the lemma.

Theorem 3. If \( \sum_{k=1}^{\infty} u_k \) is summable \( A \) to \( U' \) and \( \sum_{k=1}^{\infty} v_k \) summable \( B \) to \( V' \), and if

\[
\left| b_n v_1 + \cdots + b_1 v_n \right| < M,
\]

then \( \sum_{k=1}^{\infty} w_k \) is summable \( C \) to \( U' V' \).

Proof. Consider the triangular matrix definition

\[
a_{nk} = \frac{A_k b_{n-k+1}}{A_1 b_n + \cdots + A_n b_1}.
\]

This definition satisfies the three conditions of the lemma, for

(4) \( \frac{A_k b_{n-k+1}}{\sum_{l=1}^{n} A_l b_{n-l+1}} < \frac{A_k b_{n-k+1}}{\sum_{l=k}^{n} A_l b_{n-l+1}} < \frac{b_{n-k+1}}{\sum_{l=k}^{n} b_{n-l+1}} \);

(5) \( \sum_{k=1}^{n} |a_{nk}| = 1 \);

(6) \( A' = C'B' \),

where \( A' \), \( B' \), and \( C' \) are triangular matrix definitions with

\[
a'_{nk} = a_{n,n-k+1}, \quad b'_{nk} = \frac{b_k}{B_n}, \quad c'_{nk} = \frac{a_{n-k+1} B_k}{\sum_{l=1}^{n} A_l b_{n-l+1}}.
\]

The definition \( C' \) is regular. The theorem follows immediately from this lemma.

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