ON SOME EXTREMAL PROPERTIES OF TRIGONOMETRIC POLYNOMIALS WITH REAL ROOTS

BY J. GERONIMUS

1. Introduction. L. Fejér [1],† O. Szász [2], [3], [4] and E. v. Egerváry [5] have found many interesting extremal properties of non-negative trigonometric polynomials. In particular, Szász [3] has found that for every non-negative trigonometric polynomial of order \( n \) with real coefficients,

\[
G_n(\theta) = 1 + \Re \sum_{k=1}^{n} \gamma_k e^{ik\theta}, \quad (\gamma_k = \alpha_k + i\beta_k; \ k = 1, 2, \ldots, n),
\]

the inequality

\[
|\gamma_k| \leq 2 \cos \frac{\pi}{\left\lfloor \frac{n}{k} \right\rfloor + 2}, \quad (k = 1, 2, \ldots, n),
\]

is valid.‡

The object of this note is to find the minimum of the modulus of the first coefficient \( \gamma_n \), supposing that all roots of \( G_n(\theta) \) are real. The first problem of this kind has been considered by Blumenthal [6]; we shall return in §4 to his problem and its generalization.

2. Equality of Roots of \( G_n(\theta) \) for Problem 1. Consider the following problem.

PROBLEM 1. Find the minimum of the modulus of the first coefficient \( \gamma_n \) of a non-negative trigonometric polynomial

\[
G_n(\theta) = 1 + \Re \sum_{k=1}^{n} \gamma_k e^{ik\theta}
\]
of order \( n \) with real roots.

† Numbers in brackets refer to the Bibliography at the end.
‡ \( \Re z \) means real part of \( z \); \( [a] \) means the greatest integer \( \leq a \).
In order to solve this problem we shall prove the following simple lemma.

**Lemma 1.** All roots of the polynomial $G_n^*(\theta)$ for which the minimum in Problem 1 is attained must be equal.

Consider a non-negative trigonometric polynomial $f = \sum_{k=0}^{m} c_k e^{ik\theta}$, where $\sum_{k=0}^{m} c_k e^{ik\theta}$ is a non-negative trigonometric polynomial of order $n-2$ with real roots,

$$F_{n-2}(\theta) = R \sum_{k=0}^{n-2} \gamma_k e^{ik\theta} = \sum_{k=0}^{n-2} |\gamma_k^*| \cos (k\theta - \alpha_k),$$

where $\alpha_k = \arg \gamma_k^*$, $(k = 0, 1, 2, \ldots, n-2)$, and $\alpha_0 = 0$.

On putting $\alpha = (\theta_1 + \theta_2)/2$, $\delta = (\theta_1 - \theta_2)/2$, we see easily that

$$\gamma_0 = \gamma_0^* \left(1 + \frac{1}{2} \cos 2\delta\right) - |\gamma_1^*| \cos (\alpha_1 - \alpha) \cos \delta$$

$$+ \frac{1}{4} \left|\gamma_2^*\right| \cos (\alpha_2 - 2\alpha), \quad |\gamma_n| = \frac{1}{4} \left|\gamma_{n-2}^*\right|.$$

We see that $|\gamma_n|$ does not depend on $\alpha$, nor on $\delta$; on the other hand $\gamma_0$ is maximal for $\delta = 0$ if $\cos (\alpha_1 - \alpha) \leq 0$, or for $\delta = \pi$ if $\cos (\alpha_1 - \alpha) \geq 0$. In both cases the minimal value of $|\gamma_n|$ under condition $\gamma_0 = 1$ corresponds to $\delta = 0$ or $\delta = \pi$; therefore $\theta_1$ and $\theta_2$ coincide.‡

3. **Polynomials for which $\gamma_n$ has Extremal Values.** It follows from this lemma that $G_n^*(\theta)$ is

$$G_n^*(\theta) = C \left[1 + \cos (\theta + \alpha)\right]^n,$$

$\alpha$ being an arbitrary real argument; it may be written thus:§

$$G_n^*(\theta) = \frac{C}{2^{n-1}} \left\{\frac{1}{2} C_{2n,n} + \sum_{k=1}^{n} C_{2n,n-k} \cos k(\theta + \alpha)\right\}.$$

† It is clear that all real roots of a non-negative trigonometric polynomial are of even multiplicity.
‡ $\theta_1$ and $\theta_1 + 2\pi$ are not considered as different.
§ See [6], p. 392.
For this polynomial we have
\[ \gamma_0 = \frac{1}{2^n} C(C_{2n,n}), \quad |\gamma_n| = \frac{1}{2^{n-1}} C, \]
whence we find the ratio

\[ |\gamma_n| = \frac{2}{C_{2n,n}}. \quad (8) \]

We have proved the following theorem.

**Theorem 1.** If \( G_n(\theta) \) is a non-negative trigonometric polynomial,

\[ G_n(\theta) = 1 + \Re \sum_{k=1}^{n} \gamma_k e^{i k \theta}, \]

of order \( n \) with real roots, then

\[ \frac{2}{C_{2n,n}} \leq |\gamma_n| \leq 1; \quad (9) \]

the maximum is attained for the polynomial†

\[ G_{\text{max}}(\theta) = 1 + \cos n(\theta + \alpha), \quad (10) \]

and the minimum for the polynomial

\[ G_{\text{min}}(\theta) = \frac{2^n}{C_{2n,n}} \left( 1 + \cos (\theta + \alpha) \right)^n, \quad (11) \]

\( \alpha \) being an arbitrary real argument.

4. The Generalized Extremal Problem. Consider now the following extremal problem.

**Problem 2.** Find the minimum of the ratio

\[ \frac{A_m^2 + B_m^2}{\lambda A_0^2 + \sum_{k=1}^{m} (A_k^2 + B_k^2)}, \quad (12) \]

† See [1], [2].
where
\[ g_m(\theta) = A_0 + A_1 \cos \theta + B_1 \sin \theta + \cdots + A_m \cos m\theta + B_m \sin m\theta \]
is a trigonometric polynomial of order \( m \) with real roots, and \( \lambda \) is an arbitrary non-negative number.

The above mentioned problem of Blumenthal [6] corresponds to \( \lambda = 1 \). It is easy to see that for \( \lambda = 2 \) Problem 2 is a particular case of the Problem 1. Indeed we see that

\[
g_m^2(\theta) = G_n(\theta) = \mathcal{R} \sum_{k=0}^{n} \gamma_k e^{ik\theta}
\]
is a non-negative trigonometric polynomial of order \( n = 2m \), while

\[
\gamma_0 = A_0^2 + \frac{1}{2} \sum_{k=1}^{m} (A_k^2 + B_k^2), \quad |\gamma_n| = \frac{A_m^2 + B_m^2}{2};
\]
therefore we have for \( \lambda = 2 \)

\[
\frac{A_m^2 + B_m^2}{2A_0^2 + \sum_{k=1}^{m} (A_k^2 + B_k^2)} \geq \frac{2}{C_{2n,n}} = \frac{2}{C_{4m,2m}}.
\]

To solve our problem for all \( \lambda \geq 0 \) we shall put it in the following form.

**Problem 2'. Find the maximum of the expression**

\[
L(G) = \frac{1}{2\pi} \int_0^{2\pi} G_n(\theta) d\theta + \varepsilon \left\{ \frac{1}{2\pi} \int_0^{2\pi} [G_n(\theta)]^{1/2} d\theta \right\}^2,
\]

\( (\varepsilon \geq -1) \),

where \( G_n(\theta) \) is a non-negative trigonometric polynomial

\[
G_n(\theta) = \mathcal{R} \sum_{k=0}^{n} \gamma_k e^{ik\theta}, \quad (|\gamma_n| = 1),
\]
of order \( n = 2m \) with real roots.

5. **Equality of Roots of** \( G_n^*(\theta) \) **for Problem 2'.** We shall prove the following lemma.
Lemma 2. All roots of the polynomial $G_n^*(\theta)$ for which the maximum in Problem 2' is attained must be equal.

Put
\[
\left[ G_n(\theta) \right]^{1/2} = 2 \sin \frac{\theta - \theta_1}{2} \sin \frac{\theta - \theta_2}{2} F_{m-1}(\theta)
\]
(17)
\[
= \left[ \cos \delta - \cos (\theta - \alpha) \right] F_{m-1}(\theta),
\]
where $\alpha = (\theta_1 + \theta_2)/2$, $\delta = (\theta_1 - \theta_2)/2$, and $F_{m-1}(\theta)$ is a non-negative trigonometric polynomial of order $m-1$,

(18)
\[
F_{m-1}(\theta) = \Re \sum_{k=0}^{m-1} \tilde{c}_k e^{ik\theta},
\]
with real roots. Thus we get

(19)
\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ G_n(\theta) \right]^{1/2} d\theta = c_0 \cos \delta - \frac{1}{2} |c_1| \cos (\beta_1 - \alpha),
\]
where $\beta_k = \arg c_k$, $(k = 0, 1, \ldots, m-1)$, and $\beta_0 = 0$. Further let

(20)
\[
F_{m-1}^2(\theta) = \Re \sum_{k=0}^{2m-2} \tilde{c}_k^* e^{ik\theta},
\]
then we obtain

(21)
\[
\frac{1}{2\pi} \int_0^{2\pi} G_n(\theta)d\theta = c_0^* \left( 1 + \frac{1}{2} \cos 2\delta \right)
\]
\[
- |c_1^*| \cos (\beta_1^* - \alpha) \cos \delta + \frac{1}{4} |c_2^*| \cos (\beta_2^* - 2\alpha),
\]
where $\beta_k^* = \arg c_k^*$, $(k = 0, 1, \ldots, 2m-2)$, and $\beta_0^* = 0$. Using (19) and (21) we have

(22)
\[
L(G) = A \cos 2\delta + B \cos \delta + C,
\]
where

\[
A = \frac{1}{2} (c_0^* + \epsilon c_0^2),
\]
\[
B = - |c_1^*| \cos (\beta_1^* - \alpha) - \epsilon c_0 |c_1| \cos (\beta_1 - \alpha),
\]
\[
C = \frac{1}{4} |c_2^*| \cos (\beta_2^* - 2\alpha)
\]
\[
+ \frac{1}{4} \epsilon |c_1|^2 \cos^2 (\beta_1 - \alpha) + c_0^* + \frac{1}{2} \epsilon c_0^2.
\]
It is important to point out that $|\gamma_n| = (1/4)|c^*_{2m-2}|$ does not depend on $\alpha$, nor on $\delta$. Since we have

$$c^*_0 = c_0^* + \frac{1}{2} \sum_{k=1}^{m-1} |c_k|^2,$$

it is clear that for $\epsilon \geq -1$ we have

$$A = \frac{1}{2} (1 + \epsilon)c_0^* + \frac{1}{4} \sum_{k=1}^{m-1} |c_k|^2 > 0.$$

Therefore $L(G_n)$ is maximal for $\delta = 0$ if $B \geq 0$, and for $\delta = \pi$ if $B \leq 0$; in both cases $\theta_1$ and $\theta_2$ coincide, which proves our lemma.

6. Polynomials having the Extremal Property. We see that the polynomial $G^*_n(\theta)$ is

$$G^*_n(\theta) = 2^{n-1} [1 + \cos (\theta + \alpha)]^n$$

$$= \frac{1}{2} C_{2n,n} + \sum_{k=1}^{n} C_{2n,n-k} \cos k(\theta + \alpha),$$

and we have for it

$$L(G^*_n) = \frac{1}{2} (C_{2n,n} + \epsilon(C_{n,n/2})^2).$$

Thus we have proved the following theorem.

**Theorem 2.** If $G_n(\theta)$ is a non-negative trigonometric polynomial of order $n = 2m$,

$$G_n(\theta) = \Re \sum_{k=0}^{n} \gamma_k e^{ik\theta}, \quad |\gamma_n| = 1,$$

with real roots, then

$$\frac{1}{2\pi} \int_0^{2\pi} G_n(\theta)d\theta + \epsilon \left\{ \frac{1}{2\pi} \int_0^{2\pi} [G_n(\theta)]^{1/2}d\theta \right\}^2$$

$$\leq \frac{1}{2} (C_{2n,n} + \epsilon(C_{n,n/2})^2), \quad (\epsilon \geq -1);$$

the maximum is attained for the polynomial
\( G_n^*(\theta) = 2^{n-1} \{1 + \cos (\theta + \alpha)\}^n, \)

\( \alpha \) being an arbitrary real argument.

This result may also be stated as the following theorem.

**Theorem 2'.** If \( g_m(\theta) \) is a trigonometric polynomial of order \( m \),

\[ g_m(\theta) = A_0 + A_1 \cos \theta + B_1 \sin \theta + \cdots + A_m \cos m\theta + B_m \sin m\theta, \]

with real roots, then

\[
\begin{align*}
\frac{A_m^2 + B_m^2}{\lambda A_0^2 + \sum_{k=1}^{m} (A_k^2 + B_k^2)} & \geq \frac{2}{C_{4m,2m} + \frac{\lambda - 2}{2} (C_{2m,m})^2}, \quad (\lambda \geq 0); \\
\end{align*}
\]

this minimum is attained for the polynomial

\[ g_m^*(\theta) = C \{1 + \cos (\theta + \alpha)\}^m. \]

**Bibliography**


**Mathematical Institute, Kharkow, U.S.S.R.**