CONGRUENCES WITH A COMMON MIDDLE ENVELOPE*

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1. Introduction. Let \( C \) and \( \overline{C} \) be two rectilinear congruences whose corresponding rays \( l \) and \( \overline{l} \) are parallel; and let \( M \) be the point on the unit sphere \( S \) at which the normal is parallel to \( l \) and \( \overline{l} \). We refer the sphere to any isothermal system and take the linear element in the form \( ds^2 = e^{2\lambda}(du^2 + dv^2) \). Relative to the moving trihedral at \( M \), whose \( x \) axis is chosen tangent to the curve \( v = \text{const.} \), the coordinates of the points in which \( l \) and \( \overline{l} \) pierce the \( xy \) plane will be denoted by \( (a, b) \) and \( (\tilde{a}, \tilde{b}) \), respectively. Distances on \( l \) and \( \overline{l} \) will be measured from these points, and the positive direction will be that which corresponds to the outward-drawn normal at \( M \).

It is the purpose of this note to consider such pairs of congruences as \( C \) and \( \overline{C} \) when they have a common middle envelope, that is, when the distances to the middle points on \( l \) and \( \overline{l} \) are equal.

2. Condition that \( C \) and \( \overline{C} \) have a Common Middle Envelope. A necessary and sufficient condition that \( C \) and \( \overline{C} \) have a common middle envelope is that

\[
\frac{\partial a}{\partial u} + \frac{\partial b}{\partial v} + ar_1 - br + 2\xi = \frac{\partial \tilde{a}}{\partial u} + \frac{\partial \tilde{b}}{\partial v} + \tilde{a}r_1 - \tilde{b}r + 2\xi
\]

This may be written

\[
\frac{\partial}{\partial u}(a - \tilde{a}) + \frac{\partial}{\partial v}(b - \tilde{b}) + (a - \tilde{a}) \frac{\partial \lambda}{\partial u} + (b - \tilde{b}) \frac{\partial \lambda}{\partial v} = 0,
\]

which, upon multiplication by \( e^\lambda \), becomes

\[
\frac{\partial}{\partial u} [e^\lambda (a - \tilde{a})] = -\frac{\partial}{\partial v} [e^\lambda (b - \tilde{b})];
\]

* Presented to the Society, February 23, 1935.
‡ Foster, loc. cit., p. 163, equation (17).
hence

\[ a - \dot{a} = e^{-\lambda} \frac{\partial R}{\partial v}, \quad b - \dot{b} = -e^{-\lambda} \frac{\partial R}{\partial u}, \]

where \( R \) is an arbitrary function of \( u \) and \( v \). From (1) we have the following theorem.*

**Theorem 1.** A necessary and sufficient condition that the congruences \( C \) and \( \overline{C} \) have a common middle envelope is that the congruence defined by the point \((a - \dot{a}, b - \dot{b})\) has for its middle envelope a point, namely, the center of \( S \).

3. Rotated Congruences. Let \( C \) be the congruence defined by \((a, b)\); and let this point be rotated through an angle \( \pi/2 \) about the corresponding normal to the point \((-b, a)\).† If \( \overline{C} \) be the congruence defined by the point \((-b, a)\), we say \( C \) and \( \overline{C} \) constitute a pair of rotated congruences. We wish to determine those congruences \( C(a, b) \), which with \( \overline{C}(-b, a) \), have a common middle envelope. From (1) we must have

\[ a + \dot{b} = e^{-\lambda} \frac{\partial R}{\partial v}, \quad b - a = -e^{-\lambda} \frac{\partial R}{\partial u}. \]

The solution of these simultaneous equations will obviously give us the required condition:

\[ a = \frac{e^{-\lambda}}{2} \left( \frac{\partial R}{\partial v} + \frac{\partial R}{\partial u} \right), \quad b = \frac{e^{-\lambda}}{2} \left( \frac{\partial R}{\partial v} - \frac{\partial R}{\partial u} \right). \]

We therefore have the following result.

**Theorem 2.** A necessary and sufficient condition that a congruence \( C(a, b) \) and its rotated congruence \( \overline{C} \) have a common middle envelope is that \( a \) and \( b \) have the values given in (2).

Suppose now that \( C(a, b) \) has for its middle envelope the center of \( S \). Then \( a = e^{-\lambda}(\partial R/\partial v) \), \( b = -e^{-\lambda}(\partial R/\partial u) \). If \( C(a, b) \) be rotated to \( \overline{C}(-b, a) \), we know that \( \overline{C} \) is a normal congruence.‡

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* Foster, loc. cit., p. 173.
† The direction of rotation is immaterial.
‡ Foster, loc. cit., p. 166, Theorem 1.
Let us now consider the middle point of the line joining \((a, b)\) and \((-b, a)\); its coordinates are \([\frac{a-b}{2}, \frac{a+b}{2}]\), or

\[
\left[ \frac{e^{-\lambda}}{2} \left( \frac{\partial R}{\partial v} + \frac{\partial R}{\partial u} \right), \frac{e^{-\lambda}}{2} \left( \frac{\partial R}{\partial v} - \frac{\partial R}{\partial u} \right) \right].
\]

Since (3) is identical with (2), we have the following theorem.

**Theorem 3.** Given a square \(ABCD\), central with \(M\), which lies in the \(xy\) plane of the trihedral. If the point \(A\) defines a congruence whose middle envelope is the center of \(S\), so also does \(C\), the opposite vertex, while the opposite vertices \(B\) and \(D\) define normal congruences; and the four points which bisect the sides of the square define four congruences with a common middle envelope.

4. **\(C\) and \(\overline{C}\) Each Normal.** Let \(C\) and \(\overline{C}\) be normal congruences. Then*

\[
(4) \quad a = e^{-\lambda} \frac{\partial P}{\partial u}, \quad \bar{a} = e^{-\lambda} \frac{\partial \bar{P}}{\partial u}, \quad b = e^{-\lambda} \frac{\partial P}{\partial v}, \quad \bar{b} = e^{-\lambda} \frac{\partial \bar{P}}{\partial v}.
\]

By (1) and (4), a necessary and sufficient condition that the congruences \(C\) and \(\overline{C}\) have a common middle envelope is that

\[
(5) \quad \begin{align*}
  a - \bar{a} &= e^{-\lambda} \left( \frac{\partial P}{\partial u} - \frac{\partial \bar{P}}{\partial u} \right) = e^{-\lambda} \frac{\partial R}{\partial v}, \\
  b - \bar{b} &= e^{-\lambda} \left( \frac{\partial P}{\partial v} - \frac{\partial \bar{P}}{\partial v} \right) = -e^{-\lambda} \frac{\partial R}{\partial u}.
\end{align*}
\]

From (5), we have, from \(\partial^2 R/\partial u \partial v = \partial^2 R/\partial v \partial u\), which is the condition of integrability,

\[
\frac{\partial^2}{\partial u^2} (P - \bar{P}) + \frac{\partial^2}{\partial v^2} (P - \bar{P}) = 0.
\]

We have therefore the following theorem.

**Theorem 4.** A necessary and sufficient condition that the normal congruences \((4)\) have a common middle envelope is that \((P - \bar{P})\) be a solution of Laplace's equation.

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* Foster, loc. cit., p. 173.