THE FORM $wx + xy + yz + zu$

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1. Introduction. In the usual notation,

$$N = N[n = wx + xy + yz + zu; \ w, x, z, u > 0; \ y \geq 0]$$

denotes the number of sets $(w, x, y, z, u)$ of integers, subject to the conditions indicated, satisfying the stated equation in which $n$ is an arbitrary constant integer $>0$. Let $\zeta_r(n)$ denote the sum of the $r$th powers of all the divisors of $n$, so that $\zeta_0(n)$ is the number of divisors. Then

(1) \quad N = \zeta_2(n) - n\zeta_0(n).

This curious result is the only one of the numerous theorems on quadratic forms stated by Liouville for which (apparently) no proof has been published.*

We shall first show that (1) follows from

(2) \quad 2N' = \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n),

and then prove (2). Another similar result is stated in §5.

2. Equivalence of (1) and (2). The form in $N'$ may be written

$$yz + (z + x)u + x(w + y);$$

and hence, by the conditions on the variables, $w + y \equiv y' > y$. Thus (2) is equivalent to

(3) \quad \zeta_2(n) - 2n\zeta_0(n) + \zeta_1(n)

= $2N'[n = yz + (z + x)u + xw; \ x, y, z, w > 0; \ u \geq 0; \ w > y].$

Applying the substitution $(xz)(yw)$ to the last we see that (3) holds also when the condition $w > y$ is replaced by $w < y$.

Consider now the remaining possibility, $w = y$. The equation becomes

(4) \[ n = (x + z)(y + u); \quad x, y, z > 0; \quad u \geq 0. \]

Hence if \( n = d\delta \) is any resolution of \( n \) into a pair of positive divisors, the number of solutions of (4) for a fixed \((d, \delta)\) is

\[ N[d = x + z; x, z > 0] \times N[\delta = y + u; y > 0; u \geq 0], \]

that is, \((d-1)\delta\); and therefore the total number of solutions of (4) is \( \sum(d-1)\delta \), the sum referring to all pairs \((d, \delta)\). Thus (4) has precisely \( n\xi_0(n) - \xi_1(n) \) solutions.

But all the solutions of

(5) \[ n = yz + (z + x)u + xw; \quad x, y, z, w > 0; \quad u \geq 0 \]

are exhausted by the three mutually exclusive sets in which \( w > y, w < y, w = y \), respectively, and the number of solutions in each of these has just been determined. The total number of solutions of (5) is thus

\[ \xi_2(n) - 2n\xi_0(n) + \xi_1(n) + n\xi_0(n) - \xi_1(n), \]

or \( \xi_2(n) - n\xi_0(n) \). Hence (1) follows from (2). Conversely, by reversing the steps, (2) follows from (1), so that (1), (2) are equivalent.

3. Dependence of (2) on an Auxiliary Relation (6). Let \( \phi(u, v) \) be finite and single-valued for all integer values of the variables \( u, v \), and beyond the condition \( \phi(u, v) = -\phi(v, u) \) for integer values of \( u, v \), let \( \phi(u, v) \) be arbitrary. Then

\[ \sum \phi(w + y, z) = \sum \left[ \sum_{r=1}^{d-1} \phi(r, d) \right], \]

\( \sum \) on the left referring to all \((w, y, z)\), that on the right to all \((d, \delta)\), from

(7) \[ d\delta = n = wx + xy + yz; \quad d, \delta, w, x, y, z > 0, \]

in which all the letters denote integers and \( n \) is constant. Assuming this for the moment, we shall prove (2).

The form in (7) is invariant under the substitution \((xy)(zw)\). Hence \( \sum w = \sum z \), the sums extending over all solutions of (7). Taking \( \phi(u, v) \equiv u - v \) in (6), we get

(8) \[ 2 \sum y = \xi_2(n) - 2n\xi_0(n) + \xi_1(n), \]
the left member of which is

\[ 2 \sum N[y = y_1 + y_2; \ y_1 > 0; \ y_2 \geq 0]. \]

It follows that

\[ 2N[n = wx + (x + z)(y_1 + y_2); \ w, x, z, y_1 > 0; \ y_2 \geq 0] \]

is given by the right member of (8). By a change of notation for the variables this result is (2).

4. Proof of (6). We now prove (6). The functions \( h(u), f(u, v) \) are single-valued and finite for integer values of the variables, and beyond the conditions (for integer values of \( u \), or of \( u, v \))

\[ h(u) = h(-u), \quad f(u, v) = f(-u, v) = f(u, -v), \]

are arbitrary. Hence in a theorem concerning \( f(u, v) \) we may replace \( f(u, v) - f(v, u) \) by \( \phi(h(u), h(v)) \), where \( \phi \) is as in §3. For \( f \) we have the identity*

\[
\sum [f(d_1 + d_2, \delta_1 - \delta_2) - f(\delta_1 - \delta_2, d_1 + d_2)] \\
= \sum [\sum (d - 1)\{f(d, 0) - f(0, d)\} + 2 \sum_{r=1}^{k-1}\{f(r, d) - f(d, r)\}],
\]

the sum on the right referring to all \((d, \delta)\), that on the left to all \((d_1, \delta_1, d_2, \delta_2)\), such that

\[ d\delta = n = d_1\delta_1 + d_2\delta_2, \]

in which all letters denote integers \( >0 \) and \( n \) is constant. Hence

\[
\sum \phi(h(d_1 + d_2, \delta_1 - \delta_2)) \\
= \sum [\sum (d - 1)\phi(h(d), h(0)) + 2 \sum_{r=1}^{k-1}\phi(h(r), h(d))].
\]

In (10) take \( h(u) = |u| \). Then

\[
\sum \phi(d_1 + d_2, |\delta_1 - \delta_2|) \\
= \sum [\sum (d - 1)\phi(d, 0) + 2 \sum_{r=1}^{k-1}\phi(r, d)].
\]

* Equivalent to one stated by Liouville, Journal de Mathématiques, (2), vol. 3 (1858), pp. 282–284. The first proof, by elementary means, was given by T. Pepin, ibid., (4), vol. 4 (1888), pp. 84–92; I showed that the identity is equivalent to one between doubly periodic functions of the first and second kinds (Transactions of this Society, vol. 22 (1921), p. 215).
According as \( \delta_1 - \delta_2 > 0, \delta_1 - \delta_2 < 0, \delta_1 - \delta_2 = 0 \) we have

\[
\delta_1 = \delta_2 + \delta_1', \quad \delta_1' > 0, \quad n = d_1(\delta_2 + \delta_1') + d_2\delta_2;
\]

\[
\delta_2 = \delta_1 + \delta_2', \quad \delta_2' > 0, \quad n = d_1\delta_1 + d_2(\delta_1 + \delta_2');
\]

\[
n = \delta_1(d_1 + d_2).
\]

The third of these contributes \( \sum (d-1)\phi(d, 0) \), summed over all divisors \( d \) of \( n \), to the left of (11). The forms in the first two are equivalent under the substitution \( (d_1d_2)(\delta_1\delta_2)(\delta_1'\delta_2') \); for the first, \( |\delta_1 - \delta_2| = \delta_1' \), for the second \( |\delta_1 - \delta_2| = \delta_2' \). Hence, by the equivalence just noted, these two together contribute \( 2\sum \phi(d_1 + d_2, \delta_1') \) to the left of (11). Substituting these results into (9), (11), and changing the notation,

\[
(d_2, \delta_2, d_1, \delta_1') = (w, x, y, z),
\]

we get (6).

5. Another Similar Result. Other choices of \( \phi \) in (6) give theorems on numbers of representations. From the results already given it is easily seen that

\[
N[n = wx + xy + yz + zu + ux; \ x, y > 0; \ u, z, w \geq 0] = \xi_3(n).
\]

To prove this we require

\[
2N[n = x(w + y + u); \ x, y, w > 0; \ u \geq 0] = \xi_2(n) - \xi_1(n),
\]

which follows at once on noting that \( x = d, \ w + y + u = \delta \), where \( n = d\delta \), and that

\[
N[\delta = w + y + u; \ w, y > 0; \ u \geq 0]
\]

is the coefficient of \( q^{\delta} \) in the expansion of \( q^2(1-q^{-3}) \), and hence is \( \delta(\delta - 1)/2 \).

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