bers \( p^{n+i+m}(E_n-M) \) are all finite. Thus we have the following theorem.

**THEOREM 5.** Let \( M \) be a common boundary of three distinct domains \( D_k \), \((k = 1, 2, 3)\), such that \( D_k \) is u.l.i.-c. for \( 0 \leq i \leq n_k \), and \( n_1 \geq n_2 \geq n_3 \). Then \( n_1 + n_2 \leq n - 3 \), and if there exists \( m > 0 \) such that \( n_1 + m \leq n - 2 \) and \( n - (n_1 + m) - 1 \leq n_3 \), the Betti numbers \( p^{n+i+m}(E_n-B) \) and \( p^i(B) \), \((0 \leq i \leq n_3)\), are all finite.*

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**ON THE NORMAL RATIONAL \( n \)-IC**

**BY HELEN SCHAUCH ADAMS**

1. **Notation.** A point \( \alpha \) of \( n \)-space may be represented by the binary form \( (at)^n = (\alpha_1 t_1 + \alpha_2 t_2 )^n \) with non-symbolic coefficients \( \alpha_0, \cdots, \alpha_n \). If \( (at)^n \) is a perfect \( n \)th power \( (t_1)^n \), \( \alpha \) will be the point on \( C^n \) of \( S_n \) whose parameter is \( t_1 \), or briefly the point \( t_1 \). Also if \( (at)^n \) is a binary form, all points which satisfy the linear apolarity condition \( (\alpha a)^n = 0 \) lie on the \( S_{n-1} a \) with coordinates \( a_0, \cdots, a_n \). The \( S_{n-p} (t_1)^p (\beta t)^{n-p} \), with parameters \( \beta_0, \cdots, \beta_{n-p} \), is the osculating \( (n-p)\)-space \( O_{n-p, t_1} \) to \( C^n \) at \( t_1 \).† This notation is helpful in the development of some of the properties of the normal rational \( n \)-ic curve. Many of the analogous properties for the case \( n = 5 \) have been found by other methods by A. L. Hjelmann.‡

2. **The Axes of \( C^n \).** An axis of \( C^n \) is a line which lies in \((n-1) O_{n-1} \)'s to \( C^n \). The axes of \( C^n \) are given by

\[ (at)^n = (t_1 t)(t_2 t) \cdots (t_{n-1} t)(st) \],

parameters \( s_0, s_1, \) the \( t_i \) being parameters of points of \( C^n \).

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* Thus, although we have no actual example, it is conceivable that there exists, in \( E_5 \), a common boundary \( M \) of three domains \( D_k \) each of which is u.l.i.-c. for \( i = 0, 1 \). If so, \( p^i(D_k) \) is infinite for \( k = 1, 2, 3 \); and \( p^i(E_5-M) \) is finite.

† Grace and Young, *The Algebra of Invariants*, 1903, Chapter 11.

Any point \((zt)^n\) of the \(O_{n-1,t_1}\) determines \(n\) \(O_{n-1,1}'\)'s whose \(t\)'s are roots of \((zt)^n = 0\). Those \(t\)'s which are not \(t_1\) determine the only axis through \(Z\) not in \(O_{n-1,t_1}: (t_1t)(t_2t) \cdots (t_{n-1}t)(zt)\). This axis determines the unique image point \(Z': (t_1t)(t_2t) \cdots (t_{n-1}t)(t't)\) in \((t't)(rt)^{n-1}\), parameters \(r_0, \cdots, r_{n-1}\).

Then the points of any axis to \(C^n\) of \(S_n\) and the points of \(C^n\) are in one-to-one correspondence, and there are \(n\) axes through any point \(Z\) of an \(O_{n-1}\) only one of which lies outside \(O_{n-1}\). There is a one-to-one correspondence between the axes of \(C^n\) not in \(O_{n-1,t_1}\) and the points of \(O_{n-1,1}'\) which establishes the collineation between pairs of points in two \(O_{n-1,1}'\)'s. Similarly, there are \(\infty^{n-s-1}\) axes in an \(O_{n-1,1}'\), and the points of each of the \(\infty^{n-s}\) \(O_{n-s-1,1}'\)'s not in \(O_{n-s,t_1}\) correspond one-to-one with the points of \(O_{n-s,1}'\).

3. Axes in an \(R_i\). Any \(R_i\) meets any \(S_{2n-2j}\) in an \(S_{n-j}\) whose points correspond, by the collineation of \(\S2\), to those of an \(E_{n-j}\) in \(O_{2n-2j}\). But \(S_{n-j}\) and \(E_{n-j}\) together fix a point \(P\) of \(O_{2n-2j}\). Then in the two \(O_{2n-1}\)'s are determined two points \(P\) and \(P'\) which are images in the collineation, since the two points are intersections of corresponding \(S_{n-j}\)'s. But since such points lie on one axis, \(P\) and \(P'\) determine an axis which lies in \(R_i\). Then any \(R_i\) contains an axis of \(C^n\), and the variety of axes from an \(S_i\) which lies in an \(O_{n-1}\) is a \(V^j_{j+1}\).

4. The Osculants to \(C^n\). The \((i-1)st\) osculant to \(C^n\) at \(t_1\) is \((t_1t)^{n-i+1}(s_1t)(s_2t) \cdots (s_{i-1}t)\) as \(t_1\) varies.* Then the variety of tangents to it is

\[(t_1t)^{n-i}(s_1t)(s_2t) \cdots (s_{i-1}t)(\alpha t)\], parameters \(\alpha_0, \alpha_1\).

This variety meets the \(O_{n-i}\) \((s_1t)(s_2t) \cdots (s_{i-1}t)(s_i)(\beta t)^{n-i}\), parameters \(\beta_0, \cdots, \beta_{n-i}\) in the curve

\[C_{t_1}^{n-i}: (t_1t)^{n-i}(s_1t)(s_2t) \cdots (s_i)(s_1t)(\beta t)\], as \(t_1\) varies,

which is of the \((n-i)th\) order. Obviously the \(O_s\)'s to \(C_n^{i+1}\) form \(O_{s-1}\)'s to \(C_n^{i-1}\) and \(C_{t_1}^{n-i}\) is the \(i\)th osculant of \(C^n\) at \(t_1\).

* The first osculant is discussed by G. Castelnuovo in Studio dell'involutezione generale sulle curve razionali mediante la loro curva normale dello spazio a \(n\) dimensioni, Atti Reale Istituto Veneto, (6), vol. 4 (1885–6), p. 1173; and by St. Jolles, Die Theorie der Osculanten und des Sehnensystems der Raumcurve IV Ordnung II Species, Aachen, 1886.
Thus the $i$th osculant $C_{n-i}^{n-i}$ is the locus of the points of intersection of the tangents to $C_{n-i+1}^{n-i+1}$ after the $n-i$ lines common to the variety of the tangents and $O_{n-i}$ are removed. It is normal and rational.

5. The $n$-ahedra in an $O_{n-1,n}$ Determined by $O_{n-1}$’s from the Points of $r$. Consider any point $Y(yt)^n$ on the line $r(\alpha y+\beta z)^n$ which is not an axis, and does not meet $C^n$. The $n$ $O_{n-1}$’s from $Y$ to $C^n$ intersect the $O_{n-1,s_1}$ in the $O_{n-2}$ $(t_is_1)(\alpha t)^{n-2}$ parameters $\alpha_0, \cdots, \alpha_{n-2}$, where the $t_i$’s are roots of $(t_1y)^n=0$. Now, the $n$ $O_{n-2}$’s from $Y$ must osculate the first osculant by §4, and they form an $n$-ahedron in $O_{n-1,s_1}$. If $Z(at)^n$ is a second point of $r$, the $n$-ahedron determined by any point of $r$ will have vertices of the type

$$\prod_{j=1}^{n-1} (\alpha t_{1,i+j} + \beta r_{1,i+j})(s_1t),$$

where the $t_i$’s are roots of $(t_1y)^n=0$, and the $r_i$’s are roots of $(r_1z)^n=0$.

Let $\sum a_{ik}x_i\alpha_3\alpha_4$, $(i, k = 0, 1, \cdots, n)$, be a quadric $V_{n-2}^{n-2}$ in $O_{n-1,s_1}$. In order that any two $n$-ahedra (1) determined by $\alpha_1, \beta_1$ and $\alpha_3, \beta_2$ be self polar with respect to $V_{n-2}^{n-2}$, $n(n-1)/2 + n$ conditions must be satisfied, one more than the number required to determine $V_{n-2}^{n-2}$. Then the two $n$-ahedra can be self polar with respect to one $V_{n-2}^{n-2}$ only if the determinant $\Delta$ of all the expressions $a_{ik}x_i\alpha_3\alpha_k$ vanishes. But if $\Delta$ is arranged so that the $n(n-1)/2$ rows which express the condition that the first $n$-ahedron be self polar appear first and the $n$ rows relating to the second follow, then $\Delta$ can be so reduced that the $n$th and $(n-1)$st rows involve the same functions of the $t_i$’s, $r_i$’s, $\alpha_1$, and $\beta_1$, while the last two rows involve those same functions of the $t_i$’s, $r_i$’s, $\alpha_2$, and $\beta_2$. The differences of the elements of the two rows will then be constants in each case, and will be the same constants, so that the value of $\Delta$ is zero.

Thus the $O_{n-1}$’s from points of a line which is not an axis and does not meet $C^n$ form $n$-ahedra in an $O_{n-1,s_1}$. Obviously, the vertices of all the $n$-ahedra determined by $r$ form a locus $K^{n-1}$ which is in one-to-one correspondence with $C_t^n$ and is thus of order $n-1$. Also, there exists a single quadric variety $V_{n-2}^{n-2}$ in $O_{n-1,s_1}$ with re-
spect to which the \( n \)-ahedra determined by all points of \( r \) are self polar.

6. **Apolarities of Points Related to \( C^n \).** Let \( B(bt)^{n-1} \) and \( D(dt)^{n-1} \) be two points of \( O_{n-1,s_1} \) which are polar with respect to \( V_{n-2}^2 \). Then if \( (et)^{2n-2} \equiv (bt)^{n-1}(dt)^{n-1} \), the condition that \( B \) and \( D \) be polar is \( (ae)^{2n-2} = 0 \). But the \( (n - 1) \) \( O_{n-2} \)'s to \( C_1^t \) from \( B \) and \( D \) have points of contact which are roots of \( (tb)^{n-1} = 0 \) and \( (td)^{n-1} = 0 \), respectively. All \( 2n - 2 \) of these points are represented by \( (et)^{2n-2} = 0 \). Also the curve \( C_1^t \) meets \( V_{n-2}^2 \) in the points which are roots of \( (at)^{2n-2} = 0 \). And since \( (ae)^{2n-2} = 0 \), \( (et)^{2n-2} \) and \( (at)^{2n-2} \) are apolar.

Let \( (t_1)^{(i)}t \), \( (i = 1, \ldots, s) \), be \( s \) points of \( C^n \). The \( O_{n-1} \)'s at these points meet in an \( S_{n-s} \) containing a \( C_1^t \). Now, any of the \( \infty^{s-1} S_{n-1} \)'s through \( S_{n-s} \) meet \( C^n \) in points which are roots of

\[
\sum_{i=0}^{s} k_i (t_1^{(i)}t) (t_1)^{n-1} \equiv (my)^n = 0.
\]

Also any point \( Y \) of \( S_{n-s} \) determines points of osculation of \( O_{n-1} \)'s whose parameters are roots of \( (yt)^n = 0 \), where \( s \) of the \( t_i \)'s are the \( t_i^{(i)} \)'s. But since \( Y \) lies in all \( s \) \( O_{n-1,t_i^{(i)}} \)'s, \( (my)^n = 0 \).

Then there are the following apolarity relationships among points related to \( C^n \). The binary form representing the points of osculation of the \( O_{n-2} \)'s to \( C_1^t \) from any two points of \( O_{n-1,s_1} \) which are polar with respect to \( V_{n-2}^2 \), and the binary form representing the points of intersection of \( C_1^t \) with \( V_{n-2}^2 \) are apolar. Also, the \( \infty^{s-1} \) binary forms of the \( n \)th order representing points of intersection with \( C^n \) of \( S_{n-s} \) through the \( S_{n-s} \) of an \( s \) osculants are apolar to the binary form of the \( s \)th order representing the \( s \) points of \( C^n \) which determine the \( s \)th osculant.

7. **\( K_{t_i}^{n-1} \) Related to Two Bundles of \( S_{n-2} \)'s in \( O_{n-1,s_1} \).** The \( \infty^{n-2} S_{n-1} \)'s through \( r \) meet \( O_{n-1,t_1} \) in \( \infty^{n-2} S_{n-s} \)'s forming the bundle \( (P_{t_i}) \):

\[
(bt)(t_1)(bt)^{n-2}, \text{ parameters} \delta_0, \ldots, \delta_{n-2}.
\]

Since if \( r \) is not an axis, every \( O_{n-1} \) meets \( r \) in a point, \( O_{n-1,t_1} \) does, and the vertex of \( (P_{t_i}) \) is this point. Call it \( Y_{t_i} \). If \( (P_{r_i}) \) is a similar bundle in \( O_{n-1,r_1} \), an \( S_{n-s-1} \) of \( (P_{t_i}) \) will correspond to an \( S_{n-s-2} \) of \( (P_{r_i}) \) if they are cut out by the same \( S_{n-k} \). Then corresponding lines of \( (P_{t_i}) \) and \( (P_{r_i}) \) are
(t_1)(a)(a^{n-2}(b))$, parameters $\delta_0, \delta_1, \alpha$'s fixed,

and

$(r_1)(a)(a^{n-2}(b))$, parameters $\delta_0, \delta_1, \alpha$'s fixed.

By the projectivity of §2, $Y_t$ is a point of $K_1^{n-1}$. If a line of $O_{n-1,t}$ is defined by $S_{n-3}$'s which contain an axis, it meets its image in $O_{n-1,r}$ in the point of $K_1^{n-1}$:

$$(s_1^{(1)})(s_1^{(2)})(s_1^{(n-2)})(t_1)(r_1).$$

Then if $r$ is not an axis, $K_1^{n-1}$ is the locus of points of intersection of corresponding lines of $(P_t)$ and $(P_r)$ when the $S_{n-3}$'s defining the line of $(P_t)$ contain a fundamental axis, and the vertex of $(P_t)$ is a point of $K_1^{n-1}$. If $r$ is an axis, it meets $O_{n-1,t}$ in a point $Y$ of $K_1^{n-1}$ which corresponds in $(P_r)$ to $Y_r$, the image of $Y_t$ of $(P_t)$. Thus if $r$ is an axis, all $K_1^{n-1}$'s of $O_{n-1,t}$'s not defining $r$ are the intersections of corresponding lines of two bundles $(P_t)$ and $(P_r)$.

By the correspondence between $S_t$'s of $(P_t)$ and $(P_r)$, it can be shown that the bisecant lines, $\cdots$, the $j$-secant $S_{j-1}$'s, $\cdots$, the $(n-2)$-secant $S_{n-3}$'s of $K_1^{n-1}$ are formed by the intersections of corresponding $S_t$'s, $\cdots$, $S_j$'s, $\cdots$, $S_{n-3}$'s of $(P_t)$ and $(P_r)$.

8. The Principal $(n-2)$-ic of $S_{n-1}$ Associated with $C_n$ of $S_n$.

Let $C^n$ be the image curve of $C^n$ projected upon any $S_{n-1}$ from a vertex $S_{t-1}$ not containing points of $C^n$. It is obvious that this image curve is of the $n$th order and that it is in one-to-one correspondence with $C^n$. Obviously the image in $S_{n-1}$ of any line meeting the vertex $S_{t-1}$ is a point.

By §2, the variety of axes to $C^n$ is easily seen to be of order $2n-2$. Likewise, the order of the variety of axes meeting a line $r$ of $S_{t-1}$ which is not itself an axis, is $2n - 2$. Now from any point $Y$ of $r$ can be passed $n$ $O_{n-1}$'s to $C^n$ which determine, $n-1$ by $n-1$, $n$ axes through $Y$, and the dimensionality of the variety of axes through $r$ is therefore 2. Then the axes meeting $r$ form a surface $V_{2}^{2n-2}$. Of this surface, $r$ is an $n$-fold directrix since every point of it is $n$-fold on the surface. Now any $S_{n-1}$ through $S_{t-1}$ also passes through $r$, which counts for $n$ in its intersection with $V_{2}^{2n-2}$. The residual intersection of $S_{n-1}$ with $V_{2}^{2n-2}$ is then $n-2$ lines of $V_{2}^{2n-2}$; or every $S_{n-1}$ through $S_{t-1}$ contains $n-2$ lines of $V_{2}^{2n-2}$. It has been shown that the image of an axis through $r$
is a point; obviously the image in $S_{n-i}$ of an $S_{n-1}$ through the vertex $S_{i-1}$ is an $S_{n-i-1}$. Since there are $\infty^{n-i}$ positions of $S_{n-1}$'s through $S_{i-1}$ and $\infty^{n-i}$ $S_{n-i-1}$'s in $S_{n-i}$, * then every $S_{n-i-1}$ of $S_{n-i}$ contains $n-2$ points of the locus of images of axes through $r$. Since there are $\infty^1$ such images, the locus of the images in $S_{n-i}$ of axes through any line $r$ of the vertex of projection not an axis is a curve of the $(n-2)$nd order called a principal $(n-2)$-ic of $S_{n-i}$ associated with $C^n$. There are $\infty^{2i-4}$ such principal $(n-2)$-ics, one for each position of $r$ in $S_{i-1}$. It can be shown that if $r$ is an axis, the principal curve in $S_{n-i}$ is an $(n-3)$-ic.

9. Projection of the $K^n_{i-1}$'s of $O_{n-1,i}$. We see that $n$ of the $O_{n-2}$'s of §4 from a point $Y$ of $r$ (not an axis) intersect $n-1$ by $n-1$ in vertices of an $n$-ahedron since each is an $S_{n-1}$. Thus the vertices of such an $n$-ahedron are images of $O_{n-1,i}$ of axes to $C^n$ which meet $r$. It was also shown in §4, that these vertices lie on a $K^n_{i-1}$ of $O_{n-1,i}$. Then an $O_{n-1}$ meets the $V^n_{2n-2}$ of axes from $r$ in $K^n_{i-1}$, and $V^n_{2n-2}$ has upon it $\infty^1$ curves, $K^n_{i-1}$. Finally, the $\infty^1 K^n_{i-1}$'s of the $O_{n-1}$'s project into the principal $(n-2)$-ic of $S_{n-i}$.

10. Projection of the $n$-ahedra of $O_{n-2}$ to $C^{n-1}$, $n$ Odd. It is easily seen that there is an $n$th order involutorial relation between the points $Y$ and the vertices of the $n$-ahedra mentioned above. Then from §9 it follows that there is an $n$th order involution between the points of $r$ and those of any $K^n_{i-1}$, and that there is an involution of order $n$ defined by the points of $r$ and those of its principal $(n-2)$-ic in $S_{n-i}$, ($n$ odd). Since the $O_{n-1}$'s which intersect $O_{n-1,i}$ in faces of an $n$-ahedron discussed here meet $r$, the resulting $O_{n-2}$'s determine, with $r$, $S_{n-i}$'s which meet $S_{n-i}$ in $S_{n-i-1}$'s. Then the projection in $S_{n-i}$ of the faces of the $n$-ahedra are $S_{n-i-1}$'s which, since the original sides were $O_{n-2}$'s to $C^n_{i-1}$, have $(n-2)$-fold contact with $C^n_{i-1}$, and are inscribed in the principal $(n-2)$-ic at $n$ points of the fundamental involution † on it, $n$ odd. (When $i=2$, the $S_{n-3}$'s are stationary.)

* For the proof that $S_n$ contains $\infty^{(n-m)(m+1)} S_m$'s, ($m<n$), see G. Veronese, La superficie omaloide normale a due dimensioni e del quarto ordine dello spazio a cinque dimensioni e le sue proiezioni nel piano e nello spazio ordinario, Memorie dell'Accademia dei Lincei, (3), vol. 19 (1884), p. 347.

† For the definition and some discussion of the fundamental involution see L. Berzolari, Sulle curve razionali di uno spazio lineare ad un numero qualunque di dimensioni, Annali di Matematica, (2), vol. 21 (1893), pp. 1–25; A. Brill,
11. The Projection of an Apolarity of §6, $n$ Odd. For the $s$ points of $C^n$ mentioned in §6 may be chosen $s$ points of one group of the fundamental involution. In this case, the $S_{n-s}$ containing the $s$th osculant passes through a point $Y$ of $r$ and contains $s$ axes. By the projection into $S_{n-s-i}$, these $s$ axes determine points of one group of the fundamental involution of the principal $(n-2)$-ic from $r$. Since $S_{n-s}$ meets $r$, it projects by the usual method into an $S_{n-s-i+1}$ and the $S_{n-1}$'s through $S_{n-s}$ project into $S_{n-i-1}$'s in $S_{n-i}$. Then in $S_{n-i}$, the $s$th order binary form representing $s$ points of one group of the fundamental involution on the principal $(n-2)$-ic from $r$ is apolar to the $\infty^{s-i}$ $n$th order binary forms, each representing the points of $C^n$ lying in $S_{n-i-1}$'s through the $S_{n-s-i+1}$ containing the image of any $s$th osculant of $C^n$, $n$ odd.

12. Projected $(n-1)$-ahedra in $S_{n-i}$ Associated with the Points of $K_{t_i}^{n-1}$ in an $S_{n-2}$ of $O_{n-1,t_i}$. Since $K_{t_i}^{n-1}$ is of the $(n-1)$st order, its points are given by $(s,t)^{n-1}(dt)\equiv (st)^n$, $d$ fixed. Now any $S_{n-2}(ct)(at)^{n-2}\equiv (et)^n$, parameters $\alpha_0, \ldots, \alpha_{n-2}$, will meet $K_{t_i}^{n-1}$ in points whose parameters are the roots of $(se)^n=0$, of which there are obviously $n-1$. Then also an $S_{n-2}$ in an $O_{n-1}$ will meet $K_{t_i}^{n-1}$ in $n-1$ points.

Now by the fundamental projectivity, every point of $S_{n-2,t_i}$ is the image of an axis (§2) and since $S_{n-2,t_i}$ contains $\infty^{n-2}$ points, these axes form an $n-1$ dimensional variety. This variety is also composed of the $\infty^1$ $S_{n-2}$'s which are homologous to $S_{n-2,t_i}$ in the other $O_{n-1}$'s; that is, which are cut out of the $\infty^1 O_{n-1}$'s by the axes of the variety. Among the axes are those which meet $S_{n-2,t_i}$ in the $n-1$ points of $K_{t_i}^{n-1}$ and which must meet every other $O_{n-1}$ in $n-1$ points (in accordance with §5, $r$ is not an axis). These $n-1$ points where the axes from the points in $S_{n-2,t_i}$ of $K_{t_i}^{n-1}$ meet an $O_{n-1}$ determine that $O_{n-1}$. If, however, the $O_{n-1,\delta_i}$ is one of those defining an axis to one of these points, then the axis lies completely in $O_{n-1,\delta_i}$. In this case, the $S_{n-2}$ homologous to that of $O_{n-1,t_i}$ is defined by the $n-2$ points where the other special axes meet $O_{n-1,\delta_i}$ and the point of $r$ from which $O_{n-1,\delta_i}$ was drawn.

Since each $S_{n-2, i}$ passes through a point of $r$ which lies in the vertex $S_{i-1}$, $S_{n-2, j}$ will project into an $S_{n-i-1}$ of $S_{n-i}$. Also every $S_{n-2, i}$ contains a point of the axis which connects a point of $r$ to a point of $K_{i}^{n-1}$. Now this whole axis determines a point of $K_{i}^{n-1}$ and, since it meets $r$, must project into a point of the principal $(n-2)$-ic of $S_{n-i}$ associated with $r$. However, the point common to this axis and $S_{n-2, i}$ is on $r$ and thus projects into no portion of $S_{n-i}$. Thus the images of the other $(n-2)$ $S_{n-2, i}$'s in $S_{n-i}$ contain the same point of the principal $(n-2)$-ic of $S_{n-i}$ associated with $r$. Then, finally, the $\infty^{n-1}$ $S_{n-2, i}$'s associated with the $\infty^{n-1}$ $S_{n-1}$'s of any $O_{n-1, i}$ project into $S_{n-i}$ in $(n-1)$-ahedra whose faces are $S_{n-i}$'s, every $n-2$ of which meet in a point of the principal $(n-2)$-ic of $S_{n-i}$ associated with $r$.

13. Projection of the Variety of Axes from Points of a Line of $O_{n-1, i}$. This $V_{\infty}^2$ was defined in §3. Obviously, the lines of the variety project into lines which envelop a conic. If the line of $O_{n-1, i}$ is a bisecant of $K_{i}^{n-1}$, the conic has two points in common with the principal $(n-2)$-ic.

14. On the Image of an Axis. The points determining an axis define, $n-2$ at a time, the $(n-2)$nd osculants $C_{i_1, i_2, \ldots, i_{n-4}}$ to each of which the axis is a tangent (§4). Now every axis not meeting the vertex of projection, $S_{i-1}$, determines with it an $S_{i+1}$ which meets $S_{n-1}$ in a straight line, the image of the axis. Then the image of an axis, determined by $n-1$ points of $C^{n}$, is tangent to the $n-1$ images of the $n-1$ quadratic osculants determined by these points $n-2$ at a time.

Since the $S_{i+1}$ determined by the image of the axis and $S_{i-1}$ contains an axis (§3), any line of $S_{n-i}$ may be regarded as the projection of an axis provided $i>(n-2)/2$. The axis of which any line in $S_{n-i}$ is the image corresponds to $n-1$ points of $C^{n}$, which project into $n-1$ points of $C^{n}$, so that any line of $S_{n-1}$ corresponds to $n-1$ points of $C^{n}$.