A GENERALIZATION OF WARING'S PROBLEM*

BY L. E. DICKSON

1. Introduction. Define $g(n, m)$ so that every integer $\geq m$ is a sum of $g$ $n$th powers, while not every integer $\geq m$ is a sum of $g-1$ powers. It is customary to write $g(n)$ for $g(n, 0) = g(n, 1)$. Quite recently I evaluated $g(n)$ for every $n > 6$.

For $n = 9$ or 11, I evaluate $g(n, m)$ for each $m$ below specified large values $M$. In particular, $g(11, M) = 336$ and $g(9, M) = 163$ are small compared with $g(11) = 2132$ and $g(9) = 548$.

By use of the Hardy-Littlewood Theory and extensive tables, it was found that $g(6) \leq 160$. I here obtain $g(6) \leq 110$.

2. Asymptotic Theory. Recently I proved† that, if $n \geq 4$, every integer $\geq N$ is a sum of $s - 2 + 3k$ integral $n$th powers $\geq 0$, where the quantities are defined as follows. Let $p^\theta$ be the highest power of the prime $p$ which divides $n$. Write $\gamma = \theta + 1$ if $p > 2$, $\gamma = \theta + 2$ if $p = 2$. Let $D$ be the g.c.d. of $p-1$ and $n/p^\theta$. Write $m = D(p^\theta - 1)/(p - 1)$. Then the conditions on $s$ (for Lemma B of A) are $s \geq 2n$, $s \geq m + 1$, for every prime $p$. They hold for $s \geq 13$ if $n = 6$, and for $s \geq 2n + 1$ if $n$ is an odd prime or its square.

Employ natural logarithms. If $\vartheta(b)$ is the sum of the logarithms of all primes $\leq b$,

$$\vartheta(b) \leq \frac{6}{5} (.92129)b + 3 \log^2 b + 8 \log b + 5.$$ 

Take

$$b = (1 + n^*)^{2/(s-\delta)},$$

$$\log c = \log n + (n + 1) \log 3 + 2n^2 \log 2 + n(2n - 1)\vartheta(b),$$

$$C = 12(8n \cdot 3^{n-1})^{1/2} 2^{(n-1)/4} n^{2(n-1)/2}/(3/2)^{(1-1/n)/2},$$

$$N^{-J/n} = C/c, \quad 2J = \sigma \left(3 - \frac{1}{2n}\right) + z - \frac{(n - 1)}{2n^2},$$

* Presented to the Society, October 26, 1935.
\[ \sigma = n \left(1 - \frac{1}{n}\right)^k, \quad k > \log r / \left\{ \log n - \log (n - 1) \right\} , \]
\[ r = n^2(6n - 1)/(n - 1 - 2n^2z) , \]

where \( z \) is positive and so small that
\[ 1 + \log (3/2)^{n-1} + (n - 1)/(2n^2) \cdot \log N \leq \left( \frac{1}{3} \right)^{1/n} . \]

3. Case \( n = 6 \). The best result is obtained from \( s = 19 \). Take \( z = 0.00002 \). Then the least integer \( k \) is 31. Using base 10 except as marked, we get
\[ \log \sigma = 2.3235310, \quad -J = 0.0039946, \quad \vartheta(b) = 257.984, \]
\[ -\log c = 17086.33198, \quad \log C = 2.11363, \]
\[ \log \log N = 7.047165. \]

All integers \( < l = 2120044 \) are sums of 86 sixth powers, while all between \( l \) and \( L_0 = 51 \, 679 \, 845 \, 000 \, 000 \) are sums of 33 sixth powers. Then† all integers between \( l \) and \( L_t \) are sums of \( t + 33 \) if \( \log \log L_t = (0.0791813) + 0.9563804 \). Thus \( L_t \geq N \) if \( l = 77 \). But \( t + 33 = 110 = s - 2 + 3k \).

**Theorem 1.** Every positive integer is a sum of 110 sixth powers.

4. Case \( n = 11 \). Take \( z = 0.00001 \). The least integer \( k \) is 70. Also \( s \geq 23 \). The best result is obtained from \( k = 100, s = 43 \). Then
\[ \log \sigma = 4.90212, \quad -J = 0.0194770, \quad \vartheta(b) = 388.19352, \]
\[ -\log c = 89856.02131, \quad \log C < \frac{1}{2} n^2, \quad \log \log N = 7.343518. \]

Denote the eleventh powers of 2, \( \cdots \), 9 by \( a, \cdots, h \). By an extended algebraic table (in MS), I proved that every integer between
\[ l = 2g + h = 48 \, 560 \, 928 \, 793 \]
and \( l + 15b \) is a sum of 129 eleventh powers. By ascents, all between \( l \) and \( L_0 = 33 \, 348 \, 227 \times 10^6 \) are sums of 171, and all between \( l \) and \( L_t \) are sums of \( t + 171 \), where

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† This Bulletin, vol. 39 (1933), p. 711, Theorem 12. Here \( r = 1/6 \) to seven decimal places.
log log \( L_t = t \log 1.1 + 0.3140358 \).

Hence \( L_t \geq N \) if \( t \geq 170 \).

**Lemma 1.** Every integer \( \geq l \) is a sum of 341 eleventh powers.

Elsewhere* I proved that 302 eleventh powers suffice from \( j = 2d + e = 460 \) 453 306 to \( j + 14c \) and are necessary for \( j + 66a + 8b + 9c + 1416 \), while by ascent 321 powers suffice from \( j \) to far beyond \( l \).

**Lemma 2.** Every integer \( \geq j \) is a sum of 341 eleventh powers.

5. To obtain further results for \( n = 11 \), employ

\[
\begin{align*}
(1) \quad b &= 86a + 1019, \quad c = 23b + 38a + 1139, \\
(2) \quad d &= 11c + 15b + 16a + 808, \quad e = 7d + 5c + 14a - 11.
\end{align*}
\]

My method† to construct an algebraic table of minimum decompositions is based on leaders \( L \). Since \( L \) has a decomposition into fewer powers, the same is true of the sum of \( L \) and any linear function. The leaders \( < e \) are

\[
\begin{align*}
173a (= 10 + 2b), \quad 64a + 165b, \quad 36a + 189b, \quad 356b, \quad 253b + c, \\
75a + 38b + 10c, \quad 86a + 27c, \quad 164b + 28c, \quad 24a + 140b + 29c, \\
35a + 13b + 46c, \quad 115b + 65c, \quad 82c, \quad 126a + d, \quad 127b + d, \\
115a + 63b + 2d, \quad 87a + 87b + 2d, \quad 40a + 25b + 3d.
\end{align*}
\]

Employ linear homogeneous functions in which the coefficients of \( a, b, c, d \) are \( \geq -86, \quad -23, \quad -11, \quad -7 \), respectively, by (1) and (2), and which have no leader as component. We obtain \( 87a - b = 1029, \) 25 equations in \( a, b, c, 33 \) with \(- d, 25 \) with \(- 2d, 17 \) with \(- 3d, 14 \) with \(- 4d, 12 \) with \(- 5d, 10 \) with \(- 6d, \) and \( 8 \) with \(- 7d \).

We shall say that there is a peak \( (x, m) \), or peak \( x \) at \( m \), if \( m \) is a sum of \( x \), but not fewer, \( n \)th powers, while all integers \( > m \) are sums of fewer than \( x \) powers. Thus \( m \) is the greatest integer requiring \( x \) \( n \)th powers. If \( (x', m') \) and \( (x, m) \) are consecutive peaks, every integer between \( m' \) and \( m \) is a sum of \( x \) \( n \)th powers.

**Theorem 2.** There are exactly 24 peaks \( \geq 336 \) if \( n = 11 \). They include

\[ P_j = (1185 - 119(j-1), jc + 85a + 2047 - 120j), \quad (j = 1, 2, 3); \]
\[ R_k = (838 - 119k, (4 + k)c + 85a + 1577 - 120k), \quad (k = 0, 1, 2, 3); \]
\[ S = (954, 3c + 82a + 1697). \]

If \((x, m)\) is any of these peaks except \(R_3\), also \((x - 93, m + 22b)\) is a peak. Together, these give the peaks with \(x = 1185, 1092, 1066, 973, 954, 947, 861, 854, 838, 745, 719, 626, 600, 507, 481\). The earlier peaks are \((2132, u = 85a + 2047), (1211, u + 22b)\). The later peaks are

\[
(454, d + 4c + 9b + 5a + 529), \quad (383, 2d + 2c + 16b + 24a + 1769),
\\(373, w + 47a + 769), \quad (372, w + 75a + 740), \quad (357, w + 83a + 740),
\\(354, 3d + 2c + 17b + 36a + 762), \quad (336, 6d + 2c + 17b + 36a + 743),
\]

where \(w = 2d + 2c + 17b\).

6. Case \(n = 9\). Take \(z = 0.00001\). The least integer \(k\) is 54. Also \(s = 19\). We take \(s = 30, k = 80\), and obtain

\[ \log c = 3.8620425, \quad -J = 0.0236148, \quad \vartheta(b) = 346.26604, \]

\[ - \log e = 53104.1773, \quad \log \log \log N = 6.9443023. \]

If \(a, \ldots, e\) denote the ninth powers of \(2, 3, 6\), I proved* that all integers between \(h = 2d + e\) and \(L_0 = 58221534000\) are sums of 140 ninth powers. By ascent as in §3, all between \(h\) and \(L_t\) are sums of \(t + 140\) if

\[ \log \log L_t = (0.0511525)t + 0.3376543. \]

Then \(L_t \geq N\) if \(t = 130\). This proves the following lemma.

**Lemma 3.** Every integer \(\geq h = 2d + e\) is a sum of 270 ninth powers.

**Theorem 3.** There are exactly 19 peaks \(\geq 163\) if \(n = 9\). If \((x, m)\) is one of the four peaks \((314, c + 25a + 390), (220, c + 12b + 25a + 284), (207, 3c + 12b + 25a + 269), (194, 5c + 12b + 25a + 254)\), then \((x - 5, m + 12a)\) is a peak. The further peaks are \((548, 37a + 511), (333, 12b + 37a + 284), (208, 2c + 37a + 390), (195, 4c + 37a + 375), (182, 6c + 37a + 360), (181, d + 6b + 25a + 220), (177, d + 7b + 12a + 492), (176, d + 7b + 26a + 477), (175, d + 8b + 2a + 235), (169, d + 8b + 11a + 220), (163, 2d + 11b + 12a + 473).\]

The first four peaks give the greatest integers requiring 548, 333, 314, 309 ninth powers, respectively. There is very strong evidence that a like result holds for the next 15 peaks. For example, all integers between \(e + d\) and \(e + 2d\) are sums of 128 ninth powers; all between \(e + 2d\) and \(e + 3d\) are sums of 125.

**University of Chicago**

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**MAPS OF ABSTRACT TOPOLOGICAL SPACES IN BANACH SPACES**

**BY A. D. MICHAL AND E. W. PAXSON**

1. **Introduction.** This paper is to serve as a brief introduction to the method of considering the analysis of abstract topological spaces through the medium of homeomorphic mappings of these spaces on subsets of Banach spaces.† Our primary objective here, however, is to obtain for some general topological groups the abstract correspondents of the fundamental Lie partial differential equations for an \(r\)-parameter continuous group.‡ The essential notion is the treatment of the general situation with the aid of abstract coordinates in Banach spaces wherein the Fréchet differential may be used.§

By an abstract topological space is meant here a set of elements of completely unspecified nature, together with an undefined concept, that of neighborhood of an element (we denote the elements by small Latin letters, and the neighborhood associated with an element \(a\) by \(U(a)\)), satisfying the four Hausdorff postulates given below.||

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* Presented to the Society, November 30, 1935.
§ M. Fréchet, *Annales de l'École Normale Supérieure*, (3), vol. 42 (1925), p. 293. Briefly, \(f(x)\) on \(B_1\) to \(B_2\) has a differential at \(x = x_0\), if there exists a function \(f(x; z)\) on \(B_1^*\) to \(B_2\) linear (additive and continuous) in \(z\) and such that given a \(\rho > 0\) there is determined a \(n > 0\), so that \(|f(x_0 + z) - f(x_0) - f(x_0; z)|| \leq \rho||z||\text{ for }||z|| \leq n(\rho); f(x_0; z)\) is the differential. See also various papers by Hildebrandt, Graves, Kerner, Michal, and many others.