DEFINITION OF SUBSTITUTION

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1. Basis. The elements of this study, denoted by italic capitals, comprise atoms, at least two and perhaps infinite in number, and all finite sequences of such atoms. (The atoms are interpretable as signs, for example, and the sequences as rows of signs.) Thus each element $E$ is composed successively of possibly duplicative atoms $A_1, A_2, \cdots, A_m$, for some positive integer $m$, called the length of $E$; and an element $F$ composed successively of atoms $B_1$ to $B_n$ will be identical with $E$ if and only if $m=n$ and $A_i=B_i$ for each $i$ to $m$.

Further terminology is self-explanatory. Thus we may speak of the $k$th place of an element; this, in the case of $E$ above, is occupied by the atom $A_k$. We may speak of one element as occurring in another (as a connected segment thereof), and more particularly as occurring initially, internally, or terminally therein; of two elements as occurring overlapped in a third; of the number of occurrences of one element in another; and so on.

Juxtaposition will be used to express that binary operation of concatenation whereby any elements $E$ and $F$, composed as above, are put end to end to form that element $EF$ which is composed successively of the atoms $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n$. The element $E$ is itself describable, in terms of concatenation of its atoms, as $A_1A_2 \cdots A_m$; parentheses are suppressed, as here, in view of the obvious associativity of concatenation.

It is clear that the length of $EF$, for any $E$ and $F$, exceeds the lengths of $E$ and $F$ and equals their sum; also that $E$ occurs initially and $F$ terminally in $EF$, while $EF$ occurs neither in $E$ nor in $F$; also that atoms are of length 1, and that an element $G$ is an atom if and only if there are no elements $E$ and $F$ such that $G = EF$.

2. Substitution. The purpose of this paper is a formal definition of substitution in terms exclusively of concatenation and the following elementary logical devices: identity, applied to

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elements; the truth functions; and quantification with respect to elements. The notation will be as in *Principia Mathematica*: the sign ‘=’ for identity, the signs ‘~’, ‘⊃’, ‘· · ·’, for the truth functions, and prefixes of the forms ‘(X)’ and ‘(∃X)’ for quantification.

The proposition to be formulated is expressible verbally thus: \( W \) is the result of substituting \( X \) for \( Y \) throughout \( Z \); briefly, \( \text{sub}(W, X, Y, Z) \). When the elements are interpreted as signs and rows of signs, the notion under consideration is the notational substitution which figures so prominently in metamathematics.

In the general form in which substitution is here conceived, its formulation is complicated by the fact that the element \( Y \) for which substitution is made need not be an atom and hence may have overlapping occurrences in \( Z \). Obviously we cannot in general replace each of two overlapping occurrences of \( Y \) by \( X \), since replacement of one occurrence will mutilate the other occurrence. To this extent the notion of substitution is ambiguous. The ambiguity is resolved by stipulating that in case of overlapping occurrences left is to prevail over right; thus the result of substituting \( X \) for \( TT \) in \( TTT \) is to be \( XT \) rather than \( TX \). The result of substituting \( X \) for \( Y \) throughout \( Z \) is then describable, in general, as the result \( Z' \) of putting \( X \) for each of these occurrences of \( Y \) in \( Z \): the first (left-most); the first which begins after the end of that first; the first which begins after the end of this second; and so on. In the trivial case where \( Y \) does not occur in \( Z \), \( Z' \) is of course \( Z \).

3. *Formal Definitions.* The following abbreviations are adopted:

\[ \begin{align*}
\text{D1.} & \quad U \text{ init } V \cdot =_{dt} U = V \cdot \forall : (\exists T). \quad UT = V. \\
\text{D2.} & \quad U \text{ in } V \cdot =_{dt} U \text{ init } V \cdot \forall : (\exists T). \quad TU \text{ init } V. \\
\text{D3.} & \quad U \sim \text{ in } V \cdot =_{dt} \sim. \quad U \text{ in } V. \\
\text{D4.} & \quad \Theta(U, V, M, N) \cdot =_{dt}. \quad MUMVM in N. \quad M \sim in UV. \\
\text{D5.} & \quad \text{sub1}(U, X, Y, V) \cdot =_{dt}; U \\
& \quad X \cdot V = Y \cdot \forall : (\exists T). \quad U = TX \cdot V = TY : \sim (\exists S). \quad YS in V.
\end{align*} \]

Obviously ‘\( U \text{ init } V \)’ may be read ‘\( U \) occurs initially in \( V \)’, and
'U in V' may be read 'U occurs in V'. Again, 'sub 1(U, X, Y, V)' tells us that Y occurs in V terminally and only so and that U is the result of putting X for Y in V; this is seen as follows. For Y to occur terminally in V it is necessary and sufficient that either \( V = Y \) or \( (\exists T) \cdot V = TY \). Where \( V = Y \) it is clear further that Y does not occur in V otherwise than terminally; where \( (\exists T) \cdot V = TY \), on the other hand, in order that Y not occur in V otherwise than terminally it is obviously necessary and sufficient to add that \( \neg(\exists S) \cdot YS \text{ in } V \). In general, therefore, for Y to occur in V terminally and only so it is necessary and sufficient that

\[ V = Y \cdot \forall : (\exists T). V = TY : \sim(\exists S). YS \text{ in } V. \]

Now if \( U \) is the result of putting X for Y in V, \( U \) will be X or \( TX \) according as \( F \) is \( F \) or \( TY \). D5 thus yields the described meaning.

The definition of substitution follows:

D6. \( \text{sub}(W, X, Y, Z) = \text{df} \cdot V \sim \text{in } Z \cdot W = Z \cdot \forall \cdot (\exists G)(\exists H) : (M)(N) : (U)(V) : \text{sub} 1(U, X, Y, V). \sim \cdot V \text{ init } Z \cdot \sim \cdot \Theta(U, V, M, N) \cdot (S)(T) : \Theta(S, T, M, N) \cdot TV \text{ init } Z \cdot \sim \cdot \Theta(SU, TV, M, N) \cdot \sim \cdot W = G \cdot Z = H \cdot \forall : (\exists K). Y \sim \text{in } K \cdot W = GK \cdot Z = HK. \)

When abbreviations introduced by D1–D5 are eliminated in favor of their definiens, the definiens in D6 is seen to involve only concatenation and the elementary logical devices mentioned in §2.

4. Demonstrandum. It remains to show that D6 yields substitution in the sense of §2; that is, that \( \text{sub}(W, X, Y, Z) \), in the sense of D6, if and only if \( W \) is \( Z' \) as of §2.

Supposing \( W, X, Y, \) and \( Z \) given as constants, we define as follows:

Dt1. \( \Phi(M, N) = \text{df} \cdot (U)(V) : \text{sub} 1(U, X, Y, V) \cdot \sim \cdot V \text{ init } Z \cdot \sim \cdot \Theta(U, V, M, N) \cdot (S)(T) : \Theta(S, T, M, N) \cdot TV \text{ init } Z \cdot \sim \cdot \Theta(SU, TV, M, N). \)
Dt2. \( \Psi(F, G, H) \cdot =_{dt} (M)(N) : \Phi(M, N) \cdot \vdash \Theta(G, H, M, N) \cdot F = G \cdot Z = H \cdot \nu : (\exists K) \cdot Y \sim in K \cdot F = GK \cdot Z = HK. \)

By §1, \((U)(V) \cdot XUXVX \sim in X; by D4, then, \((U)(V) \cdot =_{\Theta(U, V, X, X)}, so that

1. \((U)(V) : V \ init Z \cdot \vdash \Theta(U, V, X, X) : =_{\sim} V \ init Z,
2. \( (G)(H) : \Phi(X, X) \cdot \vdash \Theta(G, H, X, X) : =_{\sim} \Phi(X, X),

and, trivially,

\((U)(V)(S)(T) : \Theta(S, T, X, X) \cdot TV \ init Z
3. \cdot \vdash \Theta(SU, TV, X, X). \)

By Dt1, (1), and (3),

\( \Phi(X, X) \cdot =_{\sim} (U)(V) : sub1(U, X, Y, V) \cdot \vdash \sim V \ init Z, \)
whence \( \Phi(X, X) \cdot \vdash (\exists U)(\exists V) : sub1(U, X, Y, V) \cdot V \ init Z, \)
and therefore, by D5,

\( \sim \Phi(X, X) \cdot \vdash (\exists V) : V = Y \cdot \nu : (\exists T) \cdot V = TY \vdash V \ init Z, \)
that is, \( \sim \Phi(X, X) \cdot \vdash Y \ init Z \cdot \nu : (\exists T) \cdot TY \ init Z, \)
which is to say, by D2,

4. \( \sim \Phi(X, X) \cdot \vdash Y \ in Z. \)

By Dt2, \((G)(H) : \Psi(W, G, H) \cdot \vdash \Phi(X, X) \cdot \vdash \Theta(G, H, X, X) \)
whence, by (2) and (4),

\((G)(H) : \Psi(W, G, H) \cdot \vdash Y \ in Z, \)
that is, \( (\exists G)(\exists H) : \Psi(W, G, H) \cdot \vdash Y \ in Z, \)
or, equivalently,

5. \( (\exists G)(\exists H) : \Psi(W, G, H) \cdot =_{\sim} Y \ in Z : (\exists G)(\exists H) : \Psi(W, G, H). \)

If \( Y \) does not occur in \( Z \), then, by §2, \( Z' \) is \( Z. \) Hence

6. \( Y \sim in Z \cdot W = Z \cdot =_{\sim} Y \sim in Z \cdot W = Z'. \)

By D6, Dt1, and Dt2,

\( sub(W, X, Y, Z) \cdot =_{\sim} Y \sim in Z \cdot W = Z \cdot \nu : (\exists G)(\exists H) : \Psi(W, G, H). \)
Hence, by (5) and (6),

\[
\text{sub}(W, X, Y, Z) := Y \sim \text{in } Z \cdot W = Z' \cdot \nu : Y \text{ in } Z \\
\quad : (\exists G)(\exists H) \cdot \Psi(W, G, H).
\]

(7)

Now if we can prove that

(1) \quad Y \text{ in } Z \cdot \nu : (\exists G)(\exists H) \cdot \Psi(W, G, H) \cdot \equiv \cdot W = Z',

so that \( Y \text{ in } Z : (\exists G)(\exists H) \cdot \Psi(W, G, H) \cdot \equiv \cdot Y \text{ in } Z \cdot W = Z' \), then from (7) we shall have

\[
\text{sub}(W, X, Y, Z) := Y \sim \text{in } Z \cdot W = Z' \cdot \nu : Y \text{ in } Z \cdot W = Z',
\]

that is, \( \text{sub} (W, X, Y, Z) \equiv \cdot W = Z' \) which was to be proved. It thus remains only to establish (I).

5. Proof of (I). Given that \( Y \) occurs in \( Z \), it is to be proved that

\[
(\exists G)(\exists H) \cdot \Psi(W, G, H) \cdot \equiv \cdot W = Z'.
\]

Consider the following segments \( Z_i \) of \( Z \); \( Z_1 \) extends from the beginning of \( Z \) to the end of the first occurrence of \( Y \) in \( Z \); \( Z_2 \) extends from the beginning of \( Z^1 \) to the end of the first occurrence of \( Y \) in \( Z^1 \), where \( Z^1 \) is \( Z \) deprived of its initial segment \( Z_1 \); and, in general, \( Z_{i+1} \) extends from the beginning of \( Z^i \) to the end of the first occurrence of \( Y \) in \( Z^i \), where \( Z^i \) is \( Z \) deprived of its initial segment \( Z_1 Z_2 \cdots Z_i \). By construction, \( Y \) occurs in each \( Z_i \) terminally and only so; in view of §3, then, where the \( Z^i \) are the results of putting \( X \) for \( Y \) in the respective \( Z_i \),

(8) \quad (i) \cdot \text{sub}\{Z^i, X, Y, Z_i\}.

Let \('Z_1 Z_2 \cdots Z_i'\) and \('Z^i \ Z_2 \cdots Z_i'\) be written \('Z_i!'\) and \('Z_i!'\). Thus

(9) \quad Z_1! = Z_1 \cdot Z_i! = Z_i!,

(10) \quad (i) \cdot Z_{i+1}! = Z_i! Z_{i+1} \cdot Z_i'! Z_{i+1} = Z_i'! Z_{i+1},

and, by construction,

(11) \quad (i) \cdot Z_i! \text{ init } Z.

We exhaust the segments \( Z_i \) only when we reach a point in \( Z \) beyond which \( Y \) occurs no more. Thus, where \( Z_n \) is the last of the \( Z_i \),
(12) \((K): Z = Z_n!K \cdot \mathfrak{C} \cdot Y \sim in \ K.\)

Quantification with respect to subscripts, as in (8)–(11), refers of course only to the \(n\) or fewer significant values; for example, \((i) \cdot \phi(Z_i, Z_i!'), (\exists i) \cdot \phi(Z_i, Z_i!'), \) and \((i) \cdot \phi(Z_i, Z_{i+1})'\) are short for

\[
\phi(Z_1, Z_1!) \cdot \phi(Z_2, Z_2!) \cdot \ldots \cdot \phi(Z_n, Z_n!).
\]

and

\[
\phi(Z_1, Z_2) \cdot \phi(Z_2, Z_3) \cdot \ldots \cdot \phi(Z_{n-1}, Z_n!).
\]

By §1, there are distinct atoms; let \(A\) and \(B\) be any two such, and, where \(k\) is the length of \(Z_n!Z_n!\), let \(C = ABB \ldots B\) to \(k\) occurrences of \(B\). Clearly

\[(i) \cdot C \sim in Z_n!Z_n!,\]

since the length of \(C\) is greater by one even than that of \(Z_n!Z_n!\). Now it will be shown that, for any elements \(G\) and \(H\) such that \(C \sim in GH\), there are no occurrences of \(C\) in \(CGCHC\) except the three indicated ones. First, no two occurrences of \(C\) can overlap; for, if they are distinct, one must start later than the other and hence must start at a non-initial place of the other; but all these non-initial places are occupied by \(B\), whereas \(C\) starts with \(A\). Hence if there is an occurrence of \(C\) in \(CGCHC\) other than the three indicated ones, it overlaps none of the latter; it therefore lies wholly within \(G\) or \(H\). But it cannot, since, by hypothesis, \(C \sim in GH\). Consequently there are none but the three occurrences of \(C\) in \(CGCHC\). In particular, then, it follows from (13) that there are none but the three occurrences of \(C\) in \(CZ_i!CZ_i!C\).

Since, where

\[(14) \quad D = CZ_i!CZ_i!CCZ_i!CZ_i!C \ldots CZ_n!CZ_n!C\]

\(D\) is made up wholly of segments of the form \(CZ_i!CZ_i!C\), any occurrence of \(C\) in \(D\) must lie either wholly within or partly within and partly beyond such a segment. But in the latter case the occurrence of \(C\) in question would overlap the terminal occurrence of \(C\) in \(CZ_i!CZ_i!C\), whereas we saw that no two occurrences of \(C\) could overlap. Hence every occurrence of \(C\) in \(D\) lies wholly within \(CZ_i!CZ_i!C\) for some \(i\). But \(CZ_i!CZ_i!C\) was
seen to contain only the three indicated occurrences of \( C \). Therefore \( D \) contains only the \( 3n \) occurrences of \( C \) indicated in (14).

Where \( C \sim \text{in} \ GH \), \( CGCHC \) contains, as was seen, only the three indicated occurrences of \( C \), and is hence describable as beginning and ending with \( C \) and containing just three occurrences of \( C \) and no two adjacent. But, since \( D \) contains none but its \( 3n \) indicated occurrences of \( C \), inspection of (14) shows that the only segments of \( D \) fulfilling this description of \( CGCHC \) are the segments \( CZ_i!CZ_i!C \) for the various \( i \); any other segment of \( D \) beginning and ending with \( C \) and containing three occurrences of \( C \) would contain two adjacent. Hence, if \( C \sim \text{in} \ GH \) and \( CGCHC \) \( \text{in} \ D \), \( CGCHC \) must be \( CZ_i!CZ_i!C \) for some \( i \); then \( G \) and \( H \) must be \( Z_i! \) and \( Z_i! \). Thus, in view of D4,

\[
(15) \quad (G)(H): \Theta(G, H, C, D) \Rightarrow (\exists i). \ G = Z_i! \ . \ H = Z_i! .
\]

Conversely, by (14), (13), and D4,

\[
(16) \quad (i). \ \Theta(Z_i!, Z_i!, C, D) .
\]

By (15) and (16),

\[
(17)^* \quad (G)(H): \Theta(G, H, C, D) \Rightarrow (\exists i). \ G = Z_i! \ . \ H = Z_i! .
\]

If \( \text{sub}1(U, X, Y, V) \), then, by §3, \( Y \) occurs in \( V \) terminally and only so and \( U \) is the result of putting \( X \) for \( Y \) in \( V \). But, if \( V \) contains \( Y \) just thus and if further \( Z_i!V \) \( \text{init} \ Z \), then \( V \) must be \( Z_{i+1} \), so that \( U \) becomes \( Z_{i+1} \); and where \( U \) and \( V \) are \( Z_{i+1} \) and \( Z_{i+1} \), it follows from (10) and (16) that \( \Theta(Z_i!U, Z_i!V, C, D) \). Thus

\* This exemplifies a general technique, within a concatenation system, for eliminating reference to a finite class or relation of elements in favor of reference to two properly selected elements \( C \) and \( D \). Where \( \alpha \) is the \( m \)-adic relation (or class, if \( m = 1 \)) exhibited by just the elements \( Q_{i1}, Q_{i2}, \ldots, Q_{in} \) in that order, for the various \( i \) from 1 to \( n \), and \( k \) is the length of the element

\[
Q_{i1}Q_{i2} \cdots Q_{im}Q_{i2} \cdots Q_{2m} \cdots Q_{n1}Q_{i2} \cdots Q_{nm} ,
\]

and \( C \) is \( ABB \cdots B \) to \( k \) occurrences of \( B \), and \( D \) is

\[
CQ_{i1}CQ_{i2} \cdots CQ_{im}CQ_{i2} \cdots CQ_{2m}C \cdots CQ_{n1}CQ_{i2} \cdots CQ_{nm}C ,
\]

it can be shown that

\[
CG_1CG_2 \cdots CG_mC \text{ in } D: C \sim \text{in} \ G_1G_2 \cdots G_m
\]

if and only if \( G_1, G_2, \ldots, G_m \) exhibit in that order the relation \( \alpha \). In (17) this equivalence is proved for the special case where \( m = 2; \ G, H, Z_i! \), and \( Z_i! \) answer to \( G_1, G_2, Q_{i1} \), and \( Q_{i2} \).
(U)(V)(i) : \text{sub1}(U, X, Y, V).Z_iV init Z. \Theta(Z_i'!U, Z_i!V, C, D),
and hence
(U)(V) : \text{sub1}(U, X, Y, V).:
(S)(T) : (\exists i).S = Z_i'!. T = Z_i'!. TV init Z. \Theta(SU, TV, C, D);
that is, by (17),
(U)(V) : \text{sub1}(U, X, Y, V).:
(S)(T) : \Theta(S, T, C, D). TV init Z. \Theta(SU, TV, C, D).

Again, if Y occurs in V terminally and only so, and V init Z, then V must be Z_i; thus, where sub1(U, X, Y, V) and V init Z, U and V will be Z_i' and Z_i. But, by (9) and (16), \Theta(Z_i', Z_i, C, D).
Thus

From this and (18) it follows, by Dt1, that \Phi(C, D). By Dt2, then,
(G)(H) : \Psi(W, G, H).:
(W = G . Z = H . v : (\exists K). Y \sim in K . W = GK . Z = HK,
that is, by (17),
(G)(H) : \Psi(W, G, H).:
(W = G . Z = H . v : (\exists K). Y \sim in K . W = GK . Z = HK,
and hence
(\exists G)(\exists H). \Psi(W, G, H).:
(W = Z_i'!. Z = Z_i'!. v : (\exists K). Y \sim in K . W = Z_i'!K . Z = Z_i!K.

But, whether Z = Z_i! or (\exists K). Y \sim in K. Z = Z_i!K, i must be n: for in neither case does Z contain any occurrence of Y after Z_i!.
Thus
(\exists G)(\exists H). \Psi(W, G, H).:
(W = Z_n'! . Z = Z_n'! . v :
(\exists K). W = Z_n'!K . Z = Z_n!K.

Of the occurrences of Y in Z, the one in Z_i is, by construction, the first, the one in Z_2 is the first which begins after the end of that first, and so on. Therefore if each Z_i is supplanted in Z by
$Z'_1$, so that $Z_n!$ is supplanted by $Z'_n!$, the result will be $Z'$ as described in §2. $Z'$ is thus $Z'_n!$ or $Z'_n!K$ according as $Z$ is $Z_n!$ or $Z_n!K$. By (19), then,

(20) \[ (\exists G)(\exists H). \: \Psi(W, G, H) \: \supset \: W = Z'. \]

If $\Phi(M, N)$, then, by Dt1,

\[ sub1(Z'_1, X, Y, Z_1) \: \supset \: Z_1 init Z \: \supset \: \Theta(Z'_1, Z_1, M, N) \]

and

\[ (i): \: sub1(Z'_{i+1}, X, Y, Z_{i+1}) \: \supset \: \Theta(Z'_i!, Z'_i! M, N) \]

\[ Z_iZ_{i+1} \: \supset \: \Theta(Z'_i!Z'_i+1, Z_i!Z_{i+1}, M, N). \]

But these two results reduce, in view of (8)–(11), to

\[ \Theta(Z'_1!, Z_1!, M, N) \]

and

\[ (i): \: \Theta(Z'_i!, Z'_i!, M, N) \: \supset \: \Theta(Z'_{i+1}!, Z'_{i+1}!, M, N), \]

and from these it follows that $\Theta(Z'_n!, Z_n!, M, N)$. Thus

(21) \[ (M)(N): \: \Phi(M, N) \: \supset \: \Theta(Z'_n!, Z_n!, M, N). \]

By (11) and D1, $Z$ is $Z_n!$ or else $Z_n!K$ for some $K$; and, as seen above, $Z'$ is in these respective cases $Z'_n!$ and $Z'_n!K$. Moreover, in view of (12), $Y \sim in K$. Thus

\[ Z' = Z_n! \: . \: Z = Z_n! \: . \: \forall: \: (\exists K). \: Y \sim in K \: . \: Z' = Z'_n!K \: . \: Z = Z_n!K. \]

From this and (21) it follows, by Dt2, that $\Psi(Z', Z'_n!, Z_n!)$. Hence

\[ W = Z' \: \supset \: (\exists G)(\exists H). \: \Psi(W, G, H), \]

and consequently, by (20),

\[ (\exists G)(\exists H). \: \Psi(W, G, H) \: \equiv \: W = Z', \]

which was to be proved.

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