A NOTE ON A PRECEDING PAPER*

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1. Introduction. In a paper† by the author, the following lemma was proved.

**Lemma.** If $X_r$ is an element of $A(p)$, the number $\left\{\sum^n_{q=1} \left(\frac{q}{n}\right) X_r\right\}$ is a member of the set $A\left[1 - (1 - p)^n\right]$.

Then again,‡ the author applied a theorem due to Copeland§.

It is our purpose here to extend these two theorems to apply in the field of geometrical probability. The proof of the theorem corresponding to Copeland's follows a different procedure from that given by him. As a matter of fact, the theorem of Copeland may be proved by the method given here.

2. Extension of the Lemma. The extension is as follows.

**Theorem 1.** If the numbers $(q/n)x(E_q)$, $(q = 1, 2, \ldots, n)$, are such that $x(E_q) = \phi_{E_q}(P_1), \phi_{E_q}(P_2), \ldots$, where $E_q$ is the interval $0 < y \leq \rho_q$ and $P_1, P_2, \ldots$ is a set of points admissibly ordered with respect to the function $m(E)$ (the Lebesgue measure of $E$) defined in $\Delta$; $0 < y \leq 1$, then (1) the number $\sum^M_{q=1} (q/n)x(E_q)$ is a member of the set $A\left[1 - \prod^n_{q=1}(1 - \rho_q)\right]$ and (2) the number $\sum^M_{q=1} (q/n)x(E_q)$ has the probability $\left[1 - \prod^n_{q=1}(1 - \rho_q)\right]$ where $M \leq n$.

**Proof of (1).** We know that\

(a) $\sim \sum^M_{q=1} \left(\frac{q}{n}\right) x(E_q) \sim = \prod^M_{q=1} \left(\frac{q}{n}\right) x(E_q) \cdot$

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* Presented to the Society, February 29, 1936.
† See the author's memoir The application of the theory of admissible numbers to time series with constant probability, Transactions of this Society, vol. 36 (1934), p. 517.
‡ Same reference as above, p. 524.
|| The symbol $\sum (q/n)x(E_q)$ represents the number $\{ (1/n)x(E_1) \vee \cdots \vee (M/n)x(E_M) \vee \}$, while $\Pi (q/n)x(E_q)$ represents $\{ (1/n)x(E_1) \vee \cdots \vee (M/n)x(E_M) \}$. Throughout the paper such symbols will have similar mean-
The numbers \( \sim (q/n)x(E_q) \), (\( q = 1, 2, \ldots, M \)), are independent, since \( (q/n)x(E_q) \), (\( q = 1, 2, \ldots, M \)), are independent. From (a), we obtain

\[
\sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee = \sim \prod_{q=1}^{M} \sim \left( \frac{q}{n} \right) x(E_q),
\]

and hence we have

\[
p \left[ \sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee \right] = 1 - p \left[ \prod_{q=1}^{M} \sim \left( \frac{q}{n} \right) x(E_q) \right],
\]

or

\[
p \left[ \sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee \right] = 1 - \prod_{q=1}^{M} (1 - p_q),
\]

where \( M \leq n \).

**Proof of (2).** Since

\[
\sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee = \sim \prod_{q=1}^{M} \sim \left( \frac{q}{n} \right) x(E_q),
\]

then

\[
\left( \frac{r_i}{m} \right) \left[ \sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee \right] = \sim \prod_{q=1}^{M} \sim \left( \frac{[q + (r_i - 1)n]}{mn} \right) x(E_q).
\]

Then

\[
\prod_{i=1}^{k} \left( \frac{r_i}{m} \right) \left[ \sum_{q=1}^{M} \left( \frac{q}{n} \right) x(E_q) \, \vee \right] = \prod_{i=1}^{k} \sim \prod_{q=1}^{M} \sim \left( \frac{[q + (r_i - 1)n]}{mn} \right) x(E_q).
\]

The numbers \( r_i \) are chosen such that for every set \( r_1, r_2, \ldots, r_k \), we have \( 0 < r_i \leq m \) and \( r_i \neq r_j \) if \( i \neq j \). The numbers \( \sim ([q + (r_i - 1)n]/mn)x(E_q) \) are independent. Hence the numbers \( \prod_{q=1}^{M} \sim ([q + (r_i - 1)n]/mn)x(E_q) \) are independent, from

ings. For the truth of this equality, see Copeland, *The theory of probability from the point of view of admissible numbers*, Annals of Mathematical Statistics, vol. 3 (1932), p. 149.
which it follows that the numbers $\sum_{q=1}^{M} \frac{(q+(r_i-1)n)}{mn} x(E_q)$ are independent. We may now conclude that

$$p \left\{ \prod_{i=1}^{k} \left( 1 - \frac{1}{m} \sum_{q=1}^{M} \frac{q}{n} x(E_q) \right) \right\} = \left\{ 1 - \prod_{q=1}^{M} (1 - p_q) \right\}^k.$$ 

Therefore, the number $\sum_{q=1}^{M} (q/n) x(E_q)$ is an element of $A \left[ (1 - \prod_{q=1}^{M} (1 - p_q)) \right]$, where $M \leq n$.

3. Analog of Copeland’s Theorem. In order to prove the second theorem, we shall need the following lemma.

**Lemma.** If the numbers $x_1^1, x_1^2, \ldots, x_N^1, x_2^1, x_2^2, \ldots, x_N^2, \ldots, x_1^k, x_2^k, \ldots, x_N^k$ are such that $x_i^j \cdot x_j^i = 0$, where $j \neq i$ and $x_1^1, x_1^2, \ldots, x_1^k$ are independent, it follows that the numbers $(x_1^1 \lor x_2^1 \lor \cdots \lor x_N^1), (x_1^2 \lor x_2^2 \lor \cdots \lor x_N^2), \ldots, (x_1^k \lor x_2^k \lor \cdots \lor x_N^k)$ are independent.

By hypothesis, $x_1^1, x_2^1, \ldots, x_N^1, x_1^2, x_1^2, \ldots, x_N^2$, are independent, and since $x_1^1 \cdot x_2^2 = 0$, the numbers $x_1^1 \lor x_2^2, x_1^2, x_1^2, \ldots, x_N^2$ are independent.* Then the two sets of numbers $x_1^1 \lor x_2^2, x_1^2, \ldots, x_N^2$ and $x_1^1, x_1^2, \ldots, x_N^2$ are independent, and since $(x_1^1 \lor x_2^2) \cdot x_1^1 = (x_1^1 \cdot x_2^2) \lor (x_1^1 \cdot x_2^2) = 0$, the numbers $x_1^1 \lor x_2^2 \lor x_1^2, x_1^2, \cdots, x_N^2$ are independent. In general, the numbers $x_1^1 \lor x_2^2 \lor \cdots \lor x_N^1, x_1^2, \cdots, x_N^k$ are independent.

Applying the above to each of the $(k-1)$ remaining groups of numbers, we conclude that the numbers $(x_1 \lor x_2 \lor \cdots \lor x_N^1), (x_1^2 \lor x_2^2 \lor \cdots \lor x_N^2), \ldots, (x_1^k \lor x_2^k \lor \cdots \lor x_N^k)$ are independent numbers. Hence we have proved the lemma.

We now come to the analog of the theorem of Copeland.

**Theorem 2.** If the numbers $(q/n) x(E_q)$, $(q = 1, 2, \ldots, n)$, are such that $x(E_q) = \phi_{E_q}(P_1), \phi_{E_q}(P_2), \ldots$, where $E_q$ is the interval $0 < y < \rho_q$ and $P_1, P_2, \ldots$ is admissibly ordered with respect to the function $m(E)$ defined in $A$: $0 < y \leq 1$, and if

$$X = \sum_{j=1}^{M} Y_j \lor \quad \text{and} \quad Y_j = \prod_{i=1}^{\alpha_i} \left( \frac{q_{ij}}{n} \right) x(E_{q_{ij}}), \prod_{i=\alpha_j+1}^{n} \left( \frac{q_{ij}}{n} \right) x(E_{q_{ij}}),$$

where $0 < q_{ij} \leq n$ and $q_{ij} \neq q_{ij}$ if $i \neq i'$, and where $Y_j \neq Y_j$ if $j \neq j'$, then $X$ belongs to the set $A(P)$, where

Since the numbers \((q/n)x(E_q)\) are independent, the numbers \(\sim(q/n)x(E_q)\) are independent. By hypothesis, we know that \(Y_j \cdot Y_{j'} = 0\) if \(j \neq j'\). Hence \(p(X) = P\). Now we wish to show that

\[
P\left[ \prod_{s=1}^{k} \left( \frac{r_s}{m} \right) X \right] = P^k
\]

for every positive integer \(m\) and for every set of distinct integers \(r_1, r_2, \ldots, r_k\), such that \(0 < r_s \leq m\). We know that

\[
\left( \frac{r_s}{m} \right) Y_j = \prod_{i=1}^{a_j} \left( \frac{[q_{ij} + (r_s - 1)n]}{mn} \right) x(E_{q_{ij}}) \prod_{i=a_j+1}^{n} \left( \frac{[q_{ij} + (r_s - 1)n]}{mn} \right) x(E_{q_{ij}}).
\]

The numbers constituting the above product are independent, and moreover \((r_1/m)Y_{i_1}, (r_2/m)Y_{i_2}, \ldots, (r_k/m)Y_{i_k}\) are independent regardless of whether \(j_1, j_2, \ldots, j_k\) are equal or not. Within each group any two distinct numbers are mutually exclusive; that is, \((r_s/m)Y_j \cdot (r_s/m)Y_{j'} = 0\) if \(j \neq j'\). We may now apply the above lemma. Hence the numbers

\[
\sum_{j=1}^{M} (r_1/m)Y_j \vee, \sum_{j=1}^{M} (r_2/m)Y_j \vee, \ldots, \sum_{j=1}^{M} (r_k/m)Y_j \vee
\]

are independent. We know that

\[
p\left[ \prod_{s=1}^{k} \left( \frac{r_s}{m} \right) X \right] = p\left[ \prod_{s=1}^{k} \left( \frac{r_s}{m} \right) \left\{ \sum_{j=1}^{M} Y_j \vee \right\} \right] = p\left[ \prod_{s=1}^{k} \left\{ \sum_{j=1}^{M} \left( \frac{r_s}{m} \right) Y_j \vee \right\} \right];
\]

but since the numbers \(\sum_{s=1}^{M} (r_s/m) Y_j \vee\) are independent, the last term is equal to \(P^k\). Therefore the theorem is proved.

It is obvious that the above theorems can be extended to an \(n\)-dimensional continuum.

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