[(5), (9)] \[ r \sim s. = i \]
[(4), (6)] \[ p \sim q. = r \sim s \]
[11.03] \[ (7) = (1)(2) \]
[(7), (8)] \[ (1)(2) \]
[11.2] \[ (1)(2) \sim (1) \]
[12.17] \[ (1)(2) \sim (2) \]
[(9), (10)] \[ (1) \]
[(9), (11)] \[ (2) \]

The paradox stated above is a particular case of Theorem 10, and therefore requires no further proof.

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THE BETTI NUMBERS OF CYCLIC PRODUCTS

BY R. J. WALKER

1. Introduction. In a recent paper† M. Richardson has discussed the symmetric product of a simplicial complex and has obtained explicit formulas for the Betti numbers of the two- and three-fold products. Acting on a suggestion of Lefschetz, we define a more general type of topological product and apply Richardson's methods to compute the Betti numbers of a certain one of these, the "cyclic" product.

2. Basis for $m$-Cycles of General Products. Let $S$ be a topological space and $G$ a group of permutations on the numbers $1, \cdots, n$. The product of $S$ with respect to $G$, $G(S)$, is the set of all $n$-tuples $(P_1, \cdots, P_n)$ of points of $S$, where $(P_{i_1}, \cdots, P_{i_n})$ is to be regarded as identical with $(P_1, \cdots, P_n)$ if and only if the permutation $\left\{ i_1, \cdots, i_n \right\}$ is an element of $G$. A neighborhood of $(P_1, \cdots, P_n)$ is the set of all points $(Q_1, \cdots, Q_n)$ for which $Q_i$ belongs to a fixed neighborhood of $P_i$. It is not difficult to verify that the

† M. Richardson, *On the homology characters of symmetric products*, Duke Mathematical Journal, vol. 1 (1935), pp. 50–69. We shall refer to this paper as R.
Hausdorff axioms hold for this definition of neighborhood, and hence that \( G(S) \) is a topological space. In particular, if \( G \) is the identity or the symmetric group, \( G(S) \) is, respectively, the direct or the symmetric product of \( S \). If \( G \) is the cyclic group on \( n \) elements we shall call \( G(S) \) the \( n \)-fold cyclic product of \( S \).

The space \( G(S) \) can be obtained in another manner. Let \( S^n \) denote the \( n \)-fold direct product of \( S \). Then each element \((i_1, \ldots, i_n)\) of \( G \) gives rise to an automorphism of \( S^n \) which carries \((P_1, \ldots, P_n) \) into \((P_{i_1}, \ldots, P_{i_n})\). By identifying points which are images of each other under the group of automorphisms we evidently obtain a space homeomorphic to \( G(S) \).

Now let \( K \) be a simplicial complex, \( K^n \) its direct product, and \( k = G(K) \) its product with respect to the group \( G \) of degree \( n \) and order \( r \). We then have \( r \) automorphisms \( T_\lambda \) of \( K^n \), and a continuous, single-valued transformation \( \Lambda \) of \( K^n \) into \( k \), such that \(\uparrow\)

\[
\Lambda T_\lambda = \Lambda.
\]

Richardson has shown \(\ddagger\) that \( K^n \) and \( k \) can be subdivided into simplexes in such a fashion that the transformations \( T_\lambda \) and \( \Lambda \) are simplicial. We can therefore operate with them on chains of \( K^n \). If \( E \) and \( e \) are simplexes of \( K^n \) and \( k \), respectively, such that \( e = \Lambda E \), we define the operator \( \Lambda' \) by \( \Lambda' e = \sum_\lambda T_\lambda E \). We have then

\[
\Lambda \Lambda' e = re,
\]

\[
\Lambda' \Lambda E = \sum_\lambda T_\lambda E.
\]

We also find that \( T_\lambda, \Lambda, \) and \( \Lambda' \) preserve boundaries and hence homologies.

The principal theorem of Richardson, \(\S\) concerning the Betti numbers of \( k \), is stated in terms of matrices. For actual computation we find it easier to work with the cycles themselves, and so we shall state and prove the theorem in a slightly different form.

\(\uparrow\) In the expression for the product of two transformations, the transformation represented by the right-hand symbol is to be applied first.

\(\ddagger\) R, pp. 51 and 53.

\(\S\) R, p. 52.
Theorem 1. Let \( \{ \Gamma^i \} \) be an independent basis, with respect to homology, for \( m \)-cycles, with rational coefficients, of \( K^n \), such that \( T_\lambda \Gamma^i = \pm \Gamma^\lambda \), (\( \lambda = 1, \ldots, r \)); and let \( \{ \Gamma^a \} \) be a maximal subset of \( \{ \Gamma^i \} \) such that

(a) \( T_\lambda \Gamma^a \neq \pm \Gamma^\beta \), \( (\alpha \neq \beta) \),
(b) \( T_\lambda \Gamma^a \neq -\Gamma^\alpha \),

for any \( \lambda \). Then \( \{ \gamma^a \} = \{ \Lambda \Gamma^a \} \) is an independent basis with respect to homology for the \( m \)-cycles of \( k \).

Proof. (i) The \( \gamma^a \) are independent. For suppose that we have \( \sum x_a \gamma^a \sim 0 \), that is, \( \sum x_a \Lambda \Gamma^a \sim 0 \). Then

\[
\Lambda' \sum_a x_a \Lambda \Gamma^a = \sum_a x_a \Lambda' \Lambda \Gamma^a = \sum_a x_a T_\lambda \Gamma^a \sim 0,
\]

by (3). Now if \( T_\lambda \Gamma^a = \epsilon \Gamma^i \), \( \epsilon = \pm 1 \), we cannot have \( T_\mu \Gamma^a = -\epsilon \Gamma^i \), for this would imply

\[
T_\mu^{-1} T_\lambda \Gamma^a = \epsilon T_\mu^{-1} \Gamma^i = -\epsilon \Gamma^a = -\Gamma^a,
\]

contrary to condition (b). Similarly, from (a), we cannot have \( T_\mu \Gamma^a = \pm \Gamma^\beta \), \( \beta \neq \alpha \). Hence with each such \( \Gamma^i \) there is associated an \( \epsilon_i \), a \( \Gamma^a \), and \( s_i \) values of \( \lambda \) for which \( T_\lambda \Gamma^a = \epsilon_i \Gamma^i \). If the last homology is now written in terms of the basis \( \{ \Gamma^i \} \), the coefficient of \( \Gamma^i \) will be \( \epsilon_i s_i x_a \). Since the \( \Gamma^i \) are independent, \( \epsilon_i s_i x_a = 0 \), and therefore every \( x_a = 0 \).

Use was made of the properties of the rational coefficients only in the last step of each part of the proof. Now the \( s_i \) introduced in (i) are factors of \( r \), for the \( T_\lambda \) for which \( T_\lambda \Gamma^a = \epsilon_i \Gamma^i \) evidently form a coset of the subgroup which leaves \( \Gamma^a \) invariant. It follows that the theorem will hold for any coefficient group in which each element has a unique \( r \)th part; in particular for the group of residues modulo a number prime to \( r \).

(ii) \( \{ \gamma^a \} \) is a basis. We note first that since the set \( \{ \Gamma^a \} \) is maximal every \( \Gamma^i \) is of one of the two forms \( T_\lambda \Gamma^a \) or \( \tilde{\Gamma}^i \), where for each \( j \) there is a \( \lambda_j \) such that \( T_{\lambda_j} \tilde{\Gamma}^i = -\tilde{\Gamma}^i \). Also, \( \Lambda \tilde{\Gamma}^i = \Lambda T_{\lambda_j} \tilde{\Gamma}^i = -\Lambda \tilde{\Gamma}^i \), so that \( \Lambda \tilde{\Gamma}^i = 0 \). Now if \( \gamma \) is any \( m \)-cycle of \( k \), \( \Lambda' \gamma \) is an \( m \)-cycle of \( K^n \), and so

\[
\Lambda' \gamma \sim \sum_i x_i \Gamma^i = \sum_{a, \lambda} x_a \lambda T_\lambda \Gamma^a + \sum_j x_j \tilde{\Gamma}^i.
\]
Hence

\[ \Lambda \Lambda' \gamma = r \gamma \sim \sum_{\alpha, \lambda} x_{\alpha \lambda} \Lambda T_{\lambda} \Gamma^\alpha + \sum_j x_j \Lambda \tilde{\Gamma}^j = \sum_{\alpha, \lambda} x_{\alpha \lambda} \gamma^\alpha, \]

by (2) and (1). That is,

\[ \gamma \sim \sum_{\alpha, \lambda} \frac{x_{\alpha \lambda}}{r} \gamma^\alpha, \]

3. Betti Numbers of Cyclic Products. Keeping the notation as before, we let \( G \) be the cyclic group on \( n \) elements. To compute the \( m \)-th Betti number of the cyclic product \( k \) we must count the number of \( m \)-cycles \( \Gamma^\alpha \). A basis of the type \( \{ \Gamma^i \} \) used in the theorem is obtained by taking all cycles of the form

\[ C_{m_1} \times \cdots \times C_{m_n}, \quad m_1 + \cdots + m_n = m, \]

\( C_{m_i} \) being a member of a basis of \( m_i \)-cycles of \( K \).

Following Richardson’s procedure, we obtain

\[ T_{\lambda}(C_{m_1} \times \cdots \times C_{m_\lambda} \times C_{m_{\lambda+1}} \times \cdots \times C_{m_n}) = (-1)^{\epsilon_{m_{\lambda+1}}} \cdots \times C_{m_n} \times C_{m_1} \times \cdots \times C_{m_\lambda}, \]

where

\[ \epsilon_1 = m_1m_2 + \cdots + m_1m_n = m_1(m - m_1) = mm_1 - m_1^2 \]

\[ \equiv mm_1 - m_1 \pmod{2} \]

\[ = (m - 1)m_1, \]

and by induction

\[ \epsilon_\lambda \equiv (m - 1)(m_1 + \cdots + m_\lambda) \pmod{2}. \]

Let \( q \) be a factor of \( n \), \( n = qs \), and consider all \( \Gamma^i \) which are invariant, to within change of sign, under \( G_q \), the cyclic subgroup of \( G \) of order \( q \). They necessarily have the form

\[ \Gamma_q = (C_{m_1} \times \cdots \times C_{m_s}) \times (C_{m_1} \times \cdots \times C_{m_s}) \times \cdots \times (C_{m_1} \times \cdots \times C_{m_s}), \]

there being \( q \) identical sets of factors. We must have \( q(m_1 + \)

\[ \dagger \text{S. Lefschetz, } \textit{Topology}, \text{ p. 228.} \]
that is, to have a $\Gamma_q$, $q$ must be a factor of $m$ and hence of $(m, n)$, the highest common factor of $m$ and $n$. If $t$ is a proper multiple of $q$ and a factor of $(m, n)$, it is easily seen that a $\Gamma_t$ is also a $\Gamma_q$. We denote by $\Gamma_q^*$ any $\Gamma_q$ which is not such a $\Gamma_t$, and by $A_{m,q}$ the number of $\Gamma_q^*$. The total number of $\Gamma_q$ is then $\sum A_{m,t}$, the summation being over all values of $t$ which are multiples of $q$ and factors of $(m, n)$. But the number of $\Gamma_q$ is evidently equal to the number of possible combinations of the form $C_{m_1} \times \cdots \times C_{m_s}, m_1 + \cdots + m_s = m/q$, and this is exactly $R_{m/q}(K^*)$. Hence

$$\sum A_{m,t} = R_{m/q}(K^{m/q}),$$

and from these equations we can obtain the $A_{m,q}$ step by step starting with $q = (m, n)$, or directly by the use of the Dedekind inversion formula.

Now

$$T_q \Gamma_q = (-1)^{(m-1)(m+m_1+\cdots+m_s)} \Gamma_q = (-1)^{(m-1)m/q} \Gamma_q,$$

and so if $m$ is even and $m/q$ is odd, $\Gamma_q$ is a cycle of the type $\bar{T}t^j$ of Theorem 1 and is not counted among the $P_a$. We therefore put

$$B_{m,q} = \begin{cases} 0, & \text{if } m \text{ is even and } m/q \text{ is odd,} \\ A_{m,q} & \text{otherwise.} \end{cases}$$

Consider the $s$ cycles $\Gamma_q^*, T_1 \Gamma_q^*, \cdots, T_{s-1} \Gamma_q^*$. If any two of these are equal, say $T_i \Gamma_q^* = T_j \Gamma_q^*$, ($i > j$), then $\Gamma_q^*$ is invariant, to within change of sign, under the subgroup generated by $T_i^{-1}T_i = T_{i-j}$, and hence under the minimal subgroup containing $G_q$ and $T_{i-j}$. Since $i-j<s$, $T_{i-j}$ is not an element of $G_q$ and therefore this subgroup is a $G_t$ with $t$ a proper multiple of $q$, contrary to the definition of $\Gamma_q^*$. It follows that there are exactly $s=n/q$ distinct transforms of each of the $B_{m,q}$ cycles $\Gamma_q^*$, and so we can pick out $(q/n)B_{m,q}$ of the $\Gamma_q^*$ which are not transformable into one another and which can therefore be included among the $\bar{T}t^a$ of Theorem 1. Since the cycles $\Gamma_q^*$ for different values of $q$ are not transformable into one another and since every $\Gamma^t$ is a $\Gamma_q^*$ for some $q$, we have the following result.
Theorem 2.

\[ R_m(k) = (1/n) \sum qB_{m,q}, \]

the summation being over all factors of \((m, n)\).

The following special cases may be of interest.

Corollary 1. If \(n\) is an odd prime

\[ R_m(k) = \begin{cases} 
(1/n)R_m(K^n), & \text{if } (m, n) = 1, \\
(1/n)[R_m(K^n) - R_s(K)] + R_s(K), & \text{if } m = ns.
\end{cases} \]

Corollary 2. If \(p\) is an odd prime and \(n = p^\alpha, m = p^\beta m_1, (m_1, p) = 1, \) and \(\gamma = \min \alpha, \beta, \)

\[ R_m(k) = \frac{p - 1}{n} \left[ \frac{1}{p - 1} R_m(K^n) + \sum_{i=1}^{\gamma} p^{i-1} R_m/p^i(K^{n/p^i}) \right]. \]

Corollary 3. If \(R_0(K) = 1\), then \(R_1(k) = R_1(K)\).

4. Remark. The methods used on the cyclic product can evidently be used to compute the Betti numbers of a product with respect to an arbitrary group. In general, however, the resulting formulas are too complicated to be of interest.

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