ON THE GENERATION OF THE FUNCTIONS \( C_{pq} \) AND \( Np \) OF LUKASIEWICZ AND TARSKI
BY MEANS OF A SINGLE BINARY OPERATION

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Indicating the \( n \) "truth-values" of a Lukasiewicz-Tarski logic† by the \( n \) numbers 1, 2, \ldots, \( n \), we define the functions \( C_{pq} \) and \( Np \) as follows:

\[
\begin{align*}
C_{pq} &= 1, \text{ when } p \geq q, \\
C_{pq} &= q - p + 1, \text{ when } p < q, \\
Np &= n - p + 1.
\end{align*}
\]

Thus, for example, for \( n = 3 \) we have

\[
\begin{array}{c|ccc}
C & 1 & 2 & 3 \\
\hline
1 & 1 & 2 & 3 \\
2 & 1 & 1 & 2 \\
3 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|c}
p & Np \\
\hline
1 & 3 \\
2 & 2 \\
3 & 1 \\
\end{array}
\]

I shall denote a Lukasiewicz-Tarski logic of \( n \) truth-values by \( L_n \).

In this paper I define,‡ in terms of \( C_{pq} \) and \( Np \), a function \( E_{ipq} \) such that, in each \( L_n \), \( C_{pq} \) and \( Np \) are in turn definable in terms of \( E_{i-2pq} \). The function \( E_{ipq} \) is defined by means of the following series of definitions.

**DEFINITION 1.** \( A_0p = p, A_{i+1}p = CNpA_ip \).
**DEFINITION 2.** \( B_0p = Np, B_{i+1}p = CpB_ip \).
**DEFINITION 3.** \( D_ip = CA_ipNCpNB_ip \).
**DEFINITION 4.** \( E_{i}pq = CpD_iq \).

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† For a general discussion of this logic, see Lewis and Langford, *Symbolic Logic*, pp. 199–234.
‡ D. L. Webb has recently found (The generation of any \( n \)-valued logic by one binary operation, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 252–254) a binary operation by means of which it is possible to generate any operation of any \( n \)-valued logic. His operation, however, cannot be defined in terms of \( C_{pq} \) and \( Np \) except when \( n = 2 \). This can be seen from the fact that the operations \( C_{pq} \) and \( Np \) are class-closing on the elements 1, \( n \); whereas the operation found by Webb has not this property.
In terms of $E_{pq}$ I define certain other functions as follows:

**Definition 5.** $F_{ip} = E_{i}E_{i}ppE_{i}ppE_{i}pp$.
**Definition 6.** $M_{ip} = E_{i}F_{i}p$.
**Definition 7.** $I_{ip} = E_{i}pE_{i}F_{i}qq$.

I shall now show that, in $L_n$, $M_{n-2} = Np$, and $I_{n-2}pq = Cpq$; hence that, in $L_n$, $Cpq$ and $Np$ are definable in terms of the single binary operation $E_{n-2}pq$.

**Theorem 1.** For every $n$ in $L_n$ we have

$$A_{n-2}n = n, \quad \text{and} \quad A_{n-2}p = 1 \text{ for } p \neq n.$$  

**Proof.** I prove the first part of the theorem by mathematical induction on $i$. By Definition 1, $A_{0}n = n$. Suppose that $A_{k}n = n$; then $A_{k+1}n = CnA_{k}n = Cnn = Cln = n$. Hence for every $i$ we have $A_{i}n = n$; so, in particular, $A_{n-2}n = n$.

I prove the second part of the theorem by *reductio ad absurdum*. Suppose, if possible, that the second part of the theorem is false, so that there exists a $p_0 < n$ for which $A_{n-2}p_0 > 1$. I first show that, on this supposition, $A_{ip_0} > 1$ for every $i \leq n - 2$; for if we had $A_{ip_0} = 1$ we should have $A_{i+1}p_0 = CNp_0A_{i}p_0 = CNp_01 = 1$, so we should have $A_{n-2}p_0 = 1$, contrary to hypothesis. It can be shown that $A_{1}p_0 \leq n - 2$; for from $p_0 < n$ follows $p_0 \leq n - 1$, whence $2p_0 \leq 2n - 2$, whence $2p_0 - n \leq n - 2$; and, since $A_{1}p_0 \neq 1$, we have $A_{1}p_0 = CNp_0p_0 = p_0 - (Np_0) + 1 = p_0 - (n - p_0 - 1) = 2p_0 - n$. It can also be shown that for each $k$, $(n - 2 < k > 1)$, we have $A_{k+1}p_0 < A_{k}p_0$; for from $p_0 < n$ follows $n - p_0 + 1 > 1$, so $Np_0 > 1$; whence $A_{k}p_0 - Np_0 + 1 < A_{k}p_0$, and since $A_{k+1}p_0 \neq 1$, we have $A_{k+1}p_0 = A_{k}p_0 - Np_0 + 1$. Thus we have

$$A_{n-2}p_0 < A_{n-3}p_0 < \cdots < A_{2}p_0 < A_{1}p_0 \leq n - 2.$$  

Hence $$A_{n-2}p_0 \leq A_{1}p_0 - (n - 3) \leq (n - 2) - (n - 3),$$
and $A_{n-2}p_0 \leq 1$. But this is contrary to hypothesis. Hence the second part of the theorem is true.

The proof of the following theorem is similar.

**Theorem 2.** For every $n$ in $L_n$ we have

$$B_{n-2}1 = n, \quad \text{and} \quad B_{n-2}p = 1 \text{ for } p \neq 1.$$
Theorem 3. For every $n$ in $L_n$ we have

$$D_{n-2}1 = n, \quad D_{n-2}n = 1, \quad D_{n-2}p = p \text{ for } p \neq 1, n.$$ 

Proof. By Theorems 1 and 2, and the definitions of $Cpq$ and $Np$, we have $D_{n-2}1 = CA_{n-2}1NC1NB_{n-2}1 = C1NC1Nn = C1NC11 = C1n = n, \quad D_{n-2}n = CA_{n-2}nNCnB_{n-2}n = CnNCn1 = CnNn = C1n = 1$. Suppose now that $p \neq 1, n$. Then $D_{n-2}p = CA_{n-2}pNCpNB_{n-2}p = C1NCpN1 = C1NCpn = C1N(n-p+1) = C1[n-(n-p+1)+1] = C1p = p$.

Theorem 4. For every $p \neq 1$ in $L_n$, $E_{n-2}pp = 1$; and $E_{n-2}n1 = n$.

Proof. If $p \neq 1$, $n$ then, by Theorem 3, $E_{n-2}pp = CpD_{n-2}p = Cpp = 1$. If $p = n$, then $E_{n-2}pp = CnD_{n-2}n = Cn1 = 1$. If $p = 1$, finally, $E_{n-2}p = C1D_{n-2}1 = C1n = n$.

Theorem 5. For every $p$ in $L_n$, $F_{n-2}p = 1$.

Proof. If $p \neq 1$, then, by Theorem 4, we have

$$F_{n-2}p = E_{n-2}E_{n-2}ppE_{n-2}E_{n-2}ppE_{n-2}p = E_{n-2}1E_{n-2}11 = E_{n-2}1n = C1D_{n-2}n = C11 = 1.$$ 

If $p = 1$, then, again by Theorem 4,

$$F_{n-2}p = E_{n-2}E_{n-2}11E_{n-2}E_{n-2}11E_{n-2}11 = E_{n-2}1nE_{n-2}nn = E_{n-2}n1CnD_{n-2}1 = Cnn = 1.$$ 

Theorem 6. For every $p$ in $L_n$, $M_{n-2}p = Np$.

Proof. $M_{n-2}p = E_{n-2}pF_{n-2}p = E_{n-2}p1 = CpD_{n-2}1 = Cpn = Np$.

Theorem 7. For every $p$ and $q$ in $L_n$, $I_{n-2}pq = Cpq$.

Proof.

$$I_{n-2}pq = E_{n-2}pE_{n-2}F_{n-2}qq = E_{n-2}pE_{n-2}q = E_{n-2}pC1D_{n-2}q = E_{n-2}pD_{n-2}q = CpD_{n-2}D_{n-2}q.$$ 

But, by Theorem 3, we have $D_{n-2}D_{n-2}q = q$. Hence $I_{n-2}pq = Cpq$.

Thus we have shown that in each $L_n$ it is possible to define in terms of $Cpq$ and $Np$ a function, namely, $E_{n-2}pq$, in terms of which $Cpq$ and $Np$ are again definable.

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