CONTINUITY IN TOPOLOGICAL GROUPS

BY DEANE MONTGOMERY

In the theory of topological groups it is customary to make certain assumptions concerning the continuity of the product and the continuity of the inverse. It will be shown here that for certain types of group spaces less stringent assumptions than those usually made yield the ordinary assumptions as theorems.

Suppose that $G$ is a metric space whose elements form a group. If $x$ and $y$ are any two elements of $G$, the distance between them will be denoted by $d(x, y)$ and their (group) product will be denoted by $x \cdot y$ or $xy$. The inverse of $x$ will be denoted by $x^{-1}$, and the identity of the group by $e$. If $H$ is a set of elements of $G$, then $xH$, $Hx$, and $H^{-1}$ are sets in $G$ having an obvious definition. The function $xy$ is a function defined everywhere in the product space $G \times G$. It is often assumed that this function is continuous in the two variables simultaneously, but the following theorem shows that in a large class of cases the simultaneous continuity follows from continuity in each variable separately and this with no continuity restriction whatever on the inverse function. In fact it will be shown for separable groups that the continuity of the inverse also follows from the continuity of $xy$ in $x$ and $y$ separately.

**Theorem 1.** If $G$ is locally complete and the function $xy$ is continuous in each variable separately, then it is continuous in the two variables simultaneously.

Let it be noted first that it is sufficient to prove the simultaneous continuity at $(e, e)$, for if there is a discontinuity any-
where there will be one here. In order to see this let \( a_n \) be a sequence approaching \( a \) and let \( b_n \) be a sequence approaching \( b \), while \( a_n b_n \) does not approach \( ab \). Because of the left and right continuity \( a^{-1} a_n \) and \( b_n b^{-1} \) are sequences approaching \( e \), but \( a^{-1} a_n b_n b^{-1} \) does not approach \( e \) for if it did we could use first the left and then the right continuity to show that \( a_n b_n \) approaches \( ab \).

It is convenient first to prove a lemma.

**Lemma.** If \( H \) is any open set in \( G \) and \( \varepsilon \) is any positive number, then there exists an open subset \( H_1 \) of \( H \) and a positive number \( \delta \) such that for all elements \( h \) of \( H_1 \) and any element \( a \) of \( G \) the relation \( d(a, e) < \delta \) implies the relation \( d(ah, h) \leq \varepsilon \).

Let \( B_n \) denote all elements \( h \) of \( G \) such that for any \( a, \)
\[ d(a, e) < \frac{1}{n} \text{ implies } d(ah, h) \leq \varepsilon. \]
The set \( B_n \) is closed, a fact which may be seen as follows. Suppose that \( B_n \) is not closed and that \( b_m \) is a sequence of elements of \( B_n \) approaching an element \( b \) not in \( B_n \). Since \( b \) is not in \( B_n \), there is some element \( a \) such that \( d(a, e) < \frac{1}{n} \) and (1) \( d(ab, b) > \varepsilon \). For all \( b_m \), however, (2) \( d(ab_m, b_m) \leq \varepsilon \). Because \( xy \) is continuous in \( y \), \( \lim ab_m = ab \). Thus (1) and (2) are contradictory and from this contradiction it may be concluded that \( B_n \) is closed.

Because of the left continuity of \( xy \) every \( h \) in \( H \) belongs to \( B_n \) for sufficiently large \( n \). Therefore \( H \subset \bigcap_n B_n \). Since \( H \) is of the second category,\(^*\) there must be some \( n \) such that \( H \cap B_n \) is of the second category. Then \( B_n \) must be everywhere dense in some open subset \( H_1 \) of \( H \); and from the fact that \( B_n \) is closed, \( B_n \) must include all of \( H_1 \). The lemma is now demonstrated.

The proof of Theorem 1 may now be given. Let \( G \) (for uniformity denote \( G \) by \( H_0 \)) be the first open subset of \( G \) to which the lemma is applied; by this lemma there exists a positive number \( \delta_1 \) and an open subset \( H_1 \) of \( H_0 \) such that for all elements \( a \) in \( G \) and all elements \( h \) in \( H_1 \), \( d(a, e) < \delta_1 \) implies that \( d(ah, h) \leq 1 \).

Application of the lemma next to \( H_1 \) shows that there is a \( \delta_2 \) and an open subset \( H_2 \) of \( H_1 \) such that for all elements \( a \) in \( G \) and all elements \( h \) in \( H_2 \), \( d(a, e) < \delta_2 \) implies \( d(ah, h) \leq 1/2 \).

\(^*\) Banach, *Théorie des Opérations Linéaires*, p. 14. By hypothesis \( H \) contains complete metric subspaces and therefore the statement follows from Banach's theorem at once.

\(^\dagger\) This denotes the intersection or point set product of \( H \) and \( B_n \).
Proceeding in this manner, we obtain for every \( n \) a \( \delta_n \) and an open subset \( H_n \) of \( H_{n-1} \) such that for all \( a \) in \( G \) and all \( h \) in \( H_n \), \( d(a, e) < \delta_n \) implies \( d(ah, h) \leq 1/n \). It may be assumed that the diameter of \( H_n \) is less than \( 1/n \), and that \( \bigcap_{n=1}^{\infty} H_n \) is complete, the last assumption being possible because \( G \) is locally complete. Under these conditions, we have \( \prod_{n=1}^{\infty} H_n = \prod_{n=1}^{\infty} H_n = h_0 \), where \( h_0 \) is some point of \( G \).

Let \( \epsilon \) be any positive number whatever. Since \( xy \) is continuous in \( x \) and since \( h_0^{-1} \) is constant, there is a number \( \delta \) such that, for any \( h \) in \( G \), \( d(h, h_0) < \delta \) implies \( d(hh_0^{-1}, h_0h_0^{-1}) = d(hh_0^{-1}, e) < \epsilon \). Let \( n \) be so large that \( 1/(2n) < \delta/2 \). By the definition of \( \delta_i \) it is true for all \( h \) in \( H_{2n} \) that \( d(a, e) < \delta_{2n} \) implies \( d(ah, h) \leq 1/(2n) \).

Now let \( S(e, \delta_{2n}) \) be the open sphere of center \( e \) and radius \( \delta_{2n} \) and let \( O = [S(e, \delta_{2n})] \cap [H_{2n} \cdot h_0^{-1}] \). The set \( O \) is open and includes \( e \). If \( b \) is an element of \( O \), \( b < hh_0^{-1} \), where \( h \) is in \( H_{2n} \). Let \( a \) be any other element of \( O \). Then \( d(ab, e) = d(ahh_0^{-1}, h_0h_0^{-1}) \).

But \( d(ah, h) \leq 1/(2n) \), and \( d(h, h_0) < 1/(2n) \). Therefore \( d(ab, e) < 1/n < \delta \), and it follows that

\[
d(ab, e) = d(ahh_0^{-1}, h_0h_0^{-1}) = d(ahh_0^{-1}, e) < \epsilon.
\]

Hence the function \( xy \) is continuous at \( (e, e) \), because for an arbitrary \( \epsilon \) there has been found an open set \( O \times O \) including \( e \times e \) such that for any element \( (a, b) \) in \( O \times O \), \( d(ab, e) < \epsilon \). By the remark immediately following Theorem 1, it is evident that the proof is now complete.

This theorem could be easily proved if \( G \) were assumed to be separable, by making use of known theorems. Since \( xy \) is continuous in each variable separately, it is of Baire class 1 in the two variables together. It therefore has points of continuity and if it has any points of continuity it is continuous everywhere, as can be seen from the remark immediately following the statement of the theorem. In the non-separable case \( xy \) is of class 1 as before but whether it has points of continuity does not follow in this case from any known theorem. It would be interesting to know whether or not the next theorem, which is proved for only the separable case, is also true in the non-separable case.

**Theorem 2.** If \( G \) is complete and separable and if \( xy \) is continuous in \( x \) and \( y \) separately, then \( x^{-1} \) is continuous.

*See Kuratowski, loc. cit., pp. 180, 189, for the relevant theorems.*
It follows from Theorem 1 that $xy$ is continuous in $x$ and $y$ simultaneously. It will now be shown that $x^{-1}$ is a function in the Baire classification. In order to do this it is sufficient to prove that if $F$ is any closed set in $G$, then $F^{-1}$ is a Borel set. Let $M$ denote the set of points $(x, y)$ of $G \times G$ such that $xy = e$. This set is closed because of the continuity of $xy$. Now let $N = (G \times F) \cap M$. The projection of this set on $G$ is $F^{-1}$. This is because $N$ contains those points of $G \times G$ which are of the form $(x, y)$, where $y$ is in $F$ and $xy = e$. Further, no two points of $N$ project into the same point so that $F^{-1}$ is the continuous (1-1) image of $N$. Since $N$ is not necessarily compact, it can not be concluded that $F^{-1}$ is closed, but under the present circumstances it can be concluded that $F^{-1}$ is a Borel set* and hence $x^{-1}$ is a Baire function.

The proof of the theorem is now completed by a lemma.

**Lemma.** If $G$ is separable and complete and if $xy$ is continuous in each variable separately and $x^{-1}$ is a Baire function, then $x^{-1}$ is continuous.

The proof of this lemma follows with little variation the proof of a theorem of Banach.† First note that it is sufficient to prove that $x^{-1}$ is continuous in the neighborhood of $e$ (see Banach). Since $x^{-1}$ is a Baire function it is continuous on a set $H$, where $G - H$ is of the first category. Let $a_n$ be a sequence of elements in $G$ approaching $e$. Since $G - H$ is of the first category $a_n^{-1}(G - H)$ is also of the first category. It follows (see Banach) that $G - H \cdot \prod (a_n^{-1}H)$ can not equal $G$ so that there is a point $a$ in $H$ and in $a_n^{-1}H$ for each $n$. Therefore $a_n a$ is in $H$. Since $a_n a$ approaches $a$ and since $x^{-1}$ is continuous on $H$, it follows that $(a_n a)^{-1} = a^{-1}a_n^{-1}$ approaches $a^{-1}$ and that $a_n^{-1}$ approaches $e$.

Theorem 2 clearly remains true if we replace the hypothesis of completeness by the hypothesis of local completeness.‡

* Kuratowski, loc. cit., p. 251.
† Banach, loc. cit., p. 23.
‡ In fact, if a space is locally complete, it may be metrized so as to be complete; but for some applications the hypothesis given is more convenient.