exp \((nW)\), \(n\) any integer. Then \(N\) is a discrete subgroup of the central of \(G\), and so (see [1], p. 12) \(G/N\) is a Lie group locally topologically isomorphic with \(G\).

But the homomorphism \(G \rightarrow G/N\) carries \(S_3\) into \(S_3/N\), which is simply isomorphic with \(G_3/N = G_3^*\). This and the corollary to Theorem 1 complete the proof.

E. Cartan [5] has shown that the universal covering group of the group of projective transformations of the line is topologically isomorphic in the large with no linear group.

**BIBLIOGRAPHY**


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**CHARACTERISTICS OF BIRATIONAL TRANSFORMS IN \(S_r\)**

BY B. C. WONG

1. **Introduction.** Consider a \(k\)-dimensional variety, \(V_k^n\), of order \(n\) in an \(r\)-space, \(S_r\). Let us project \(V_k^n\) from a general \((r-k-t-1)\)-space of \(S_r\) upon a general \((k+t)\)-space of \(S_r\) and denote the projection by \(\rho V_k^n\). We are supposing that \(1 \leq t \leq k\). Then upon \(\rho V_k^n\) lies a double variety, \(D_{k-t}\), of dimension \(k-t\) and order \(b_t\) and upon \(D_{k-t}\) lies a pinch variety, \(W_{k-t-1}\), of dimension \(k-t-1\) and order \(j_{k+t-1}\). Since the symbol \(W_{-1}\) is without meaning, we thus obtain \(2k-1\) characteristics \(b_1, b_2, \cdots, b_k, j_2, j_3, \cdots, j_k\). The symbol \(j_1\) has a meaning which will be explained subsequently.

Now let a general \((r-k+q-2)\)-space, \(S_{r-k+q-2}\), \((1 \leq q \leq k)\), be given in \(S_r\). Through this \(S_{r-k+q-2}\) pass \(k-q+1\) primes of \(S_r\) and \(k-q\) of these are tangent to \(V_k^n\). The points of contact form a \((k-q)\)-dimensional variety, \(U_{k-q}\). Denote its order by \(m_q\). Thus
we obtain \( k \) further characteristics \( m_1, m_2, \ldots, m_k \). If we project \( V_k^p \) upon a \((k+1)\)-space, \( S_{k+1} \), of \( S_r \), we see that \( m_q \) is the class of the \( V_q^p \) in which a \((q+1)\)-space of \( S_{k+1} \) meets the projected variety. We also say that \( m_q \) is the class of the \( q \)-dimensional variety in which an \((r-k+q)\)-space of \( S_r \) meets \( V_k^p \).

In the case where \( V_k^p \) is the complete intersection of \( r-k \) general primals, of orders \( n_1, n_2, \ldots, n_{r-k} \), respectively, in \( S_r \), the values of \( j, b, m \) are known* and they are

(I) \[ j_t = n_1 n_2 \cdots n_{r-k} \sum (n_1 - 1)(n_2 - 1) \cdots (n_t - 1), \]

(II) \[ b_t = \frac{1}{2} n_1 n_2 \cdots n_{r-k} [n_1 n_2 \cdots n_{r-k} - 1 - \sum (n_1 - 1) \]
\[ - \sum (n_1 - 1)(n_2 - 1) \cdots \]
\[ - \sum (n_1 - 1)(n_2 - 1) \cdots (n_t - 1)]
\[ = \frac{1}{2} n_1 n_2 \cdots n_{r-k} \left[ \sum (n_1 - 1)(n_2 - 1) \cdots (n_{t+1} - 1) \right] \]
\[ + \sum (n_1 - 1)(n_2 - 1) \cdots (n_{t+2} - 1) + \cdots \]
\[ + \sum (n_1 - 1)(n_2 - 1) \cdots (n_{r-k} - 1), \]

(III) \[ m_q = n_1 n_2 \cdots n_{r-k} \sum_h \sum (n_1 - 1)^{h_1} (n_2 - 1)^{h_2} \cdots (n_q - 1)^{h_q}, \]

where

\[ h = h_1 + h_2 + \cdots + h_q = q. \]

We shall refer to these values later.

In this paper we propose to determine the values of the same characteristics for the variety \( V_k^p \) in \( S_r \) which we consider as the birational transform of a \( k \)-dimensional variety, say \( \Phi_k^p \), of order \( v \) in a \( p \)-space, \( \Sigma_p \), for \( p < r \). We confine ourselves to the case where \( \Phi_k^p \) is the complete intersection of \( p-k \) general primals of \( \Sigma_p \), of respective orders \( \nu_1, \nu_2, \ldots, \nu_{p-k} \), given by the equations

(1) \[ F^{(1)} = 0, \ F^{(2)} = 0, \cdots, \ F^{(p-k)} = 0, \]

where $F^{(i)}$ is a homogeneous function of degree $\nu_i$ in the variables $\xi_0, \xi_1, \ldots, \xi_p$. The order of $\Phi_k^r$ is $\nu = \nu_1 \nu_2 \cdots \nu_{p-k}$. The corresponding characteristics $\eta_i, \beta_i, \mu_q$ of $\Phi_k^r$ are given by (I), (II), (III) if we replace in the right-hand members $r$ by $\rho$ and $n_i$ by $\nu_i$.

We suppose that the transformation of $\Phi_k^r$ into $V_k^n$ is accomplished by means of a general linear $\infty r$-system, $|\psi|$, without base varieties of any kind, of $(k-1)$-dimensional varieties of order $\nu N$, and that $|\psi|$ is the intersection of $\Phi_k^r$ and a general linear $\infty r$-system, $|\phi|$, of primals of order $N$, none passing through $\Phi_k^r$, given by the equation

$$a_0 \phi^{(0)} + a_1 \phi^{(1)} + \cdots + a_r \phi^{(r)} = 0,$$

the $\phi$'s being linearly independent homogeneous polynomials of degree $N$ in the $(p+1)$ $\xi$'s. Then, the order of $V_k^n$ is $n = \nu N^k = \nu_1 \nu_2 \cdots \nu_{p-k} N^k$. The coordinates of the points on $V_k^n$ are given by

$$\sigma x_0 = \phi^{(0)}(\xi_0, \xi_1, \ldots, \xi_p), \quad \sigma x_1 = \phi^{(1)}(\xi_0, \xi_1, \ldots, \xi_p),$$
$$\cdots, \quad \sigma x_r = \phi^{(r)}(\xi_0, \xi_1, \ldots, \xi_p),$$

where the $\xi$'s satisfy equations (1).

It is to be noted that an $h$-dimensional locus of order $l$ on $\Phi_k^r$ goes into an $h$-dimensional locus of order $l N^k$ on $V_k^n$. For $h = k$, $\Phi_k^r$ goes into $V_k^n$, where $n = \nu N^k$.

We shall first, in §2, derive a general relation connecting the $b$'s and the $f$'s for a general variety which has no extraordinary singular points. The determination of the values of the $m$'s of our variety $V_k^n$ will be given in §3 and the determination of those of the $j$'s in §4. The values of the $b$'s will then be obtained with the aid of the relation derived in §2. Incidentally, we find it interesting to express the $m$'s and $j$'s in terms of the $\mu$'s and $\eta$'s, respectively, of $\Phi_k^r$.

2. The Relation between the b's and the j's. Let $C^l$ be a curve of order $l$, in a space of dimension greater than 2, whose points are paired in an (irrational) involution $I_2$. Suppose that $C^l$ has $d$ actual nodes at each of which two corresponding points of $I_2$ coincide but lie on different branches of the curve. If $i$ denotes the number of simple points of $C^l$ at each of which two corresponding points of $I_2$ become united, the order of the ruled su-
face, which may be a cone, whose generators are lines joining corresponding points of the involution is, as is well known,

$$R = (2l - i - 2d)/2.$$  

Now consider a general $k$-dimensional variety $V_k$, of any order $n$, without extraordinary singular points, in $S_r$ and let it be intersected by a general $(r-k+t)$-space of $S_r$ in a $V_1$. If we project $V_1$ upon a $(2l-1)$-space, $S_{2l-1}$, we see that the projection $\Gamma V_1$ has a double curve $D_1$ of order $b_{l-1}$ and $j_1$ pinch points. This $\Gamma V_1$ may certainly be regarded as the projection of a $V_1$ in a $(2l)$-space, $S_{2l}$, the $i V_1$ being assumed to be a projection of $V_1$. Let $Z$ be the point, taken in a general position of $S_{2l}$, from which $i V_1$ is projected into $\Gamma V_1$ in $S_{2l-1}$. There are $\infty^1$ lines through $Z$ meeting $i V_1$ in two distinct points and the locus of these lines is a ruled surface, in fact a cone, of order $b_{l-1}$. This cone meets $i V_1$ in a curve $c$ of order $2b_{l-1}$, of which the double curve $D_1$ on $\Gamma V_1$ is the projection. The curve $c$ has $b_1$ actual nodes which are the improper double points of $i V_1$. There are $j_1$ elements of the cone tangent to $i V_1$ and also to $c$. The projections of the points of contact are the pinch points on $\Gamma V_1$. Now on $c$ is an involution of pairs of points set up by the elements of the cone. There are $b_1$ points each of which is the union of two corresponding points on different branches of the curve and $j_1$ points each of which is the union of two corresponding points on a simple branch of the curve. Putting $R = b_{l-1}$, $l = 2b_{l-1}$, $i = j_1$, $d = b_1$ in the relation of the paragraph just preceding, we have the desired relation

$$b_{l-1} = \frac{4b_{l-1} - j_1 - 2b_1}{2}, \text{ or}$$

(3)  

$$2b_{l-2} = j_1 + 2b_1.$$  

By letting $t = 1, 2, \cdots, k$, successively, we obtain

$$2b_0 = j_1 + 2b_1, \quad 2b_1 = j_2 + 2b_2, \quad \cdots, \quad 2b_{k-1} = j_k + 2b_k.$$  

As we shall see presently, $b_0 = n(n-1)/2$ and $j_1$ is identical with the class $m_1$ of a plane section of the projection $i V_k$ in a $(k+1)$-space. The relation (3) may be replaced by the relation

(4)  

$$2b_s - 2b_t = j_{s+1} + j_{s+2} + \cdots + j_1, \quad (s > t).$$  

Note that this relation, or relation (3), is satisfied by (I) and (II).
3. The Determination of the $m$'s. Returning to the $V_k^n$ which is the birational transform of $\Phi_k^n$ in $\Sigma_p$, we see at once that $m_q$, \(1 \leq q \leq k\), is the order of the variety $U_{k-q}$ on $V_k^n$ which has for image on $\Phi_k^n$ the complete intersection $\Theta_{k-q}$ of $\Phi_k^n$ and the Jacobian variety $\omega_{p-q}$ of the $p-k$ primals, given by (1), intersecting in $\Phi_k^n$, and any $k-q+2$ independent primals of the system $|\phi|$, say $\phi^{(1)} = 0, \phi^{(2)} = 0, \ldots, \phi^{(k-q+2)} = 0$. $\Theta_{k-q}$ is the locus of the points of contact between $\Phi_k^n$ and the $\infty k-1$ system determined by the $k-q+2$ primals just mentioned which are tangent to $\Phi_k^n$. The conditions of contact are given by

$$
\begin{vmatrix}
F_0^{(1)} & F_1^{(1)} & \cdots & F_p^{(1)} \\
\vdots & \vdots & & \vdots \\
F_0^{(p-k)} & F_1^{(p-k)} & \cdots & F_p^{(p-k)} \\
\phi_0^{(1)} & \phi_1^{(1)} & \cdots & \phi_p^{(1)} \\
\vdots & \vdots & & \vdots \\
\phi_0^{(k-q+2)} & \phi_1^{(k-q+2)} & \cdots & \phi_p^{(k-q+2)}
\end{vmatrix} = 0,
$$

where $F_i^{(h)}, \phi_i^{(h)}$ are written in place of $\partial F^{(h)}/\partial \xi_i, \partial \phi^{(h)}/\partial \xi_i$, respectively. This equality represents the Jacobian $\omega_{p-q}$. The matrix being of $p+1$ columns and $p-q+2$ rows, the order of $\omega_{p-q}$ is

$$
H_q = \sum_{i=0}^{q} \left( k + 1 - i \right) \left( N - 1 \right)^{q-i} \cdot \sum_i (\nu_1 - 1)^{h_1}(\nu_2 - 1)^{h_2} \cdots (\nu_i - 1)^{h_i},
$$

(5)

where

$$
h_1 + h_2 + \cdots + h_i = i.
$$

Then the order of $\Theta_{k-q}$ is $\nu_1\nu_2 \cdots \nu_{p-k}H_q$ and therefore the order of $U_{k-q}$ on $V_k^n$ which is the transform of $\Theta_{k-q}$ is

$$
m_q = \nu_1\nu_2 \cdots \nu_{p-k}N^{k-q}H_q.
$$

(6)

It is of interest to express $m_q$ in terms of the $\mu$'s belonging to $\Phi_k^n$. The various values of the $\mu$'s are obtained from (III) by changing $r$ to $p$ and $n_i$ to $\nu_i$. By taking account of (5), we have, from (6), writing $\mu_0$ in place of $\nu$ or $\nu_1\nu_2 \cdots \nu_{p-k}$,

4. The Determination of the \( j \)'s and the \( b \)'s. The projection 
\[ \pi_{k-1}V_k^* \]

in a \((k+t-1)\)-space of \( S_r \), of \( V_k^* \) has a pinch variety 
\( W_{k-t} \) of order \( \pi_k \). This \( W_{k-t} \) has for image on \( \Phi^* \) the complete 
intersection of \( \Phi^* \) and the variety \( \pi_{p-t} \) common to all the primals each of which is the Jacobian of \( p+1 \) of the following \( p+t \) primals:

\[
F^{(1)} = 0, \quad F^{(2)} = 0, \quad \ldots, \quad F^{(\rho-k)} = 0,
\]

\[
\phi^{(1)} = 0, \quad \phi^{(2)} = 0, \quad \ldots, \quad \phi^{(k+t)} = 0.
\]

The \( k+t \) primals represented by the \( \phi \)'s equated to zero are supposed to be independent members of the system \(|\phi|\) given by (2). The equations of \( \pi_{p-t} \) are

\[
\begin{pmatrix}
F_0^{(1)} & F_1^{(1)} & \cdots & F_\rho^{(1)} \\
\vdots & \ddots & \ddots & \vdots \\
F_0^{(\rho-k)} & F_1^{(\rho-k)} & \cdots & F_\rho^{(\rho-k)} \\
\phi_0^{(1)} & \phi_1^{(1)} & \cdots & \phi_\rho^{(1)} \\
\vdots & \ddots & \ddots & \vdots \\
\phi_0^{(k+t)} & \phi_1^{(k+t)} & \cdots & \phi_\rho^{(k+t)}
\end{pmatrix} = 0.
\]

The left-hand member being a matrix of \( \rho+t \) rows and \( \rho+1 \) columns, the order of \( \pi_{p-t} \) is\(^*\)

\[
C_t = \sum_{i=0}^{t} \binom{k+l}{i} (N-1)^i \sum (\nu_1 - 1)(\nu_2 - 1) \cdots (\nu_{t-i} - 1)
\]

\[
= \sum_{i=0}^{t} \binom{k+l}{t-i} (N-1)^{t-i} \sum (\nu_1 - 1)(\nu_2 - 1) \cdots (\nu_i - 1),
\]

\(^*\) Salmon, loc. cit.
and the order of the intersection of $\Phi _{p-t}^k$ and $\pi _{p-t}$ is $\nu C_t$. Therefore, the order of $W_{k-t}$ is

$$j_t = \nu N^{k-t} C_t$$

$$= \nu_1 \nu_2 \cdots \nu_{p-k} N^{k-t} \sum_{i=0}^{t} \binom{k+t}{t-i} (N-1)^{t-i} \cdot \sum (\nu_1 - 1)(\nu_2 - 1) \cdots (\nu_t - 1).$$

(8)

Thus, for $t=1$, we have

$$j_1 = \nu_1 \nu_2 \cdots \nu_{p-k} N^{k-1} [(k+1)(N-1) + \sum (\nu_1 - 1)] = m_1.$$

Now in terms of the $\eta$'s of $\Phi _{p-t}^k$ we have, from (1) and (8),

$$j_t = N^{k-t} \sum_{i=0}^{t} \binom{k+t}{t-i} (N-1)^{t-i} \eta_i$$

$$= N^{k-t} \left[ \binom{k+t}{t} (N-1)^{t-1} \eta_0 + \binom{k+t}{t-1} (N-1)^{t-2} \eta_{1} + \cdots \right.$$  

$$+ \binom{k+t}{1} (N-1) \eta_{t-1} + \eta_t \right],$$

where $\eta_0 = \nu = \nu_1 \nu_2 \cdots \nu_{p-k}$.

To determine the $b$'s we make use of the relation (3) or (4) of §2. A little calculation yields

$$b_t = \frac{1}{2} \nu \left[ \nu N^{2k} - \sum_{h=0}^{t} N^{k-h} C_h \right],$$

where $C_h$ is given by (7) if we replace in it $t$ by $h$. Since any $(k-t)$-dimensional variety of order $l$ on $\Phi _{p-t}^k$ goes into a $(k-t)$-dimensional variety of order $lN^{k-t}$ on $V_k^n$, we see that

$$\frac{2b_t}{N^{k-t}} = \nu^2 N^{k+t} - \nu \sum_{h=0}^{t} N^{t-h} C_h$$

is the order of the image on $\Phi _{p-t}^k$ of the double variety $D_{k-t}$ on the projection $i V_k^n$ in a $(k+t)$-space.

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