The analogs of Theorems 3 and 5 in three or more dimensions require methods other than those developed in the present note, and are postponed for another occasion.

HARVARD UNIVERSITY

ON SEQUENCES OF INDEFINITE INTEGRALS

BY M. K. GOWURIN

1. Introduction. The chief result concerning the subject of this paper is due to Lebesgue* and can be formulated as follows:

If \( \{f_n(t)\} \) is a sequence of functions defined and integrable in \( J = (0, 1) \) and if for every measurable set \( e \subset J \)

\[
\lim_{n \to \infty} \int_e f_n(t) \, dt = 0,
\]

then the sequence of indefinite integrals

\[
\int_e f_n(t) \, dt
\]

is uniformly absolutely continuous.

G. Fichtenholz† has shown that the same conclusion remains true, if the equality (1) is satisfied for all open sets \( e \).

S. Saks‡ considered the space \( R = \{x\} \) of the characteristic

\[
U(x, y) = \begin{cases} 
  e, & x > 0, \\
  1, & x = 0, \\
  e^{-1}, & x < 0.
\end{cases}
\]


‡ On some functionals, Transactions of this Society, vol. 35 (1933), pp. 549–556.
functions of measurable sets contained* in $J$, by introducing
the metric:

$$
\| x \| = \int_0^1 x(t) dt = mE_i(x(t) = 1),
$$

$$
d(x, y) = \int_0^1 | x(t) - y(t) | dt.
$$

He proved the following generalization of Lebesgue's theorem:

*If the equality (1) written in the form

$$(1') \quad \lim_n F_n(x) = \lim_n \int_0^1 f_n(t)x(t) dt = 0,$$

is satisfied for all points $x$ belonging to a certain set $H$ of the sec­
ond category in $R$, then the sequence of indefinite integrals (2) is
uniformly absolutely continuous, that is, the sequence of functionals

$$(2') \quad F_n(x) = \int_0^1 f_n(t)x(t) dt$$

is equally continuous in $R$, and $F_n(x) \to 0$ for all $x \in R$.

G. Fichtenholz has stated the following problem: *What is the
general characteristic of the sets $H$ in $R$, such that the equal con­tinuity of the sequence (2') follows from its convergence to zero†
on $H$?* I shall study in particular the role of the category of $H$
in this problem.‡

* The space $R$ is complete but not linear. The sum of two elements of $R$,
$x + y$, exists if $x(t) \cdot y(t) = 0$, their difference $x - y$ exists if $x(t) \cdot y(t) = y(t)$. (Here
as everywhere, all the equalities referring to the characteristic functions must
be satisfied almost everywhere.) Further on we shall not prove the existence
of sums and differences occurring in the text, since it is evident from the course
of reasoning. The zero in the space $R$ is the characteristic function of the vacu­
ous set. Note also the equality $\| x + y \| = \| x \| + \| y \|$. Obviously, $J$ can denote
not only an interval, but every measurable set. We shall apply this statement
in §3.

† As he showed, the convergence to zero of the sequence (2') on all the $R$
always follows from its convergence to zero on such a set $H$.

‡ [Added in proof.] As Saks has kindly pointed out to me, the totality
of all the open sets considered by Fichtenholz is of the first category in $R$.
Nevertheless, it seems to me that the example stated below is not without
interest.
Without solving the problem stated above, I give in this paper an example which shows that in any case the category $H$ is not the decisive factor in the question under consideration. Of course it is not at all difficult to construct an example, of a set $H$ which is of the first category in $R$, but, being enlarged by all the existing linear combinations of its elements, forms a set of the second category in $R$. It is more natural, however, to consider the set $H$ as being additive (that is, such that every existing linear combination of its elements belongs to it again). We shall prove the following statement.

**Theorem.** If the equality $(1')$ is satisfied for all points $x$ belonging to some spherical surface in $R$, $S^1(x_0, r)$, $(r<1)$, which does not pass through the point $0$, then the system $(2')$ is equally continuous in $R$ and $F_n(x) \to 0$ everywhere in $R$.

In general, the set $H = S(x_0, r)$ is not additive, but it is easy to see that the additive extension of $S(x_0, r)$ is nowhere dense in $R$.

2. **Proof of Theorem.** Let $x_0 = 0$. Then the spherical surface $S(0, r)$ consists of the points of norm $r$. Let us suppose that our assertion is wrong and that there exists a system $(2')$ converging to zero on $S(0, r)$, but not equally continuous. There exists an $\epsilon$ such that, for every $\delta > 0$ and every integer $N$, $R$ will contain an element $x$ such that $\|x\| < \delta$ and $|F_n(x)| > \epsilon$ for a certain $n \geq N$. Denote by $H_m$ the set of points $x$ of $S(0, r)$ such that $|F_n(x)| \leq \epsilon/6$ for $n > m$. Evidently all $H_m$ are closed and as $\sum H_m = S(0, r)$, at least one of them, for instance $H_{m_0}$, is of the second category in $S(0, r)$. Being closed, $H_{m_0}$ contains in $S(0, r)$ a sphere $T(0, r; \xi_0, \rho)$ or, more briefly, $T$ ($T$ consists of such points $x$ that $\|x\| = r$ and $d(x, \xi_0) \leq \rho$).

According to the assumption there exists in $R$ an element $x_1$ such that

$$\|x_1\| = \alpha_1 < \frac{r}{3}, \quad \alpha_1 < \frac{1 - r}{3}, \quad \alpha_1 < \frac{\rho}{4},$$

and for some $\nu > m_0$, $|F_\nu(x_1)| > \epsilon$. For example, let $F_\nu(x_1) > \epsilon$.

Consider an arbitrary element $\tilde{x}$ of the spherical surface $S(0, \alpha_1)$. Construct such an element $z$ that $z + x_1$ and $z + \tilde{x}$ exist and are contained in $T$. The method of construction de-
pends upon the quantity \( \beta = \| (x_1 + \bar{x} - x_1 \bar{x}) \xi_0 \| \), which can be equal to, greater, or less than \( \alpha_1 \).

In the first case \( z = \xi \), where \( \xi = \xi_0 - (x_1 + \bar{x} - x_1 \bar{x}) \xi_0 \).

In the second case we put \( z = \xi + \eta \), where \( \eta \) is the characteristic function of a set of measure \( \beta - \alpha_1 \), lying in \( E_t(\xi_0(t) + x_1(t) + \bar{x}(t) = 0) \). It follows from \( \beta - \alpha_1 \leq \alpha_1 \) and from

\[
mE_t(\xi_0(t) + x_1(t) + \bar{x}(t) = 0) \geq 1 - (\| \xi_0 \| + \| x_1 \| + \| \bar{x} \|) = 1 - r - 2\alpha_1 \geq 1 - r - 2 \frac{1 - r}{3} = \frac{1 - r}{3} > \alpha_1,
\]

that such a set exists.

In the third case \( z = \xi - \chi \), where \( \chi \) is the characteristic function of a set of measure \( \alpha_1 - \beta \) lying in \( E_t(\xi(t) = 1) \). It follows from

\[
mE_t(\xi(t) = 1) = \| \xi \| \geq r - 2\alpha_1 > \alpha_1,
\]

that such a set exists.

In each of these cases \( \| z \| = r - \alpha_1 \) and \( d(z, \xi_0) \leq 3\alpha_1 \). Evidently \( z \) satisfies the conditions indicated above. Then

\[
F_\sigma(z + x_1) = F_\sigma(z) + F_\sigma(x_1), \quad F_\sigma(z + \bar{x}) = F_\sigma(z) + F_\sigma(\bar{x});
\]

\[
F_\sigma(\bar{x}) = F_\sigma(z + \bar{x}) - F_\sigma(z) = F_\sigma(z + \bar{x}) - F_\sigma(z + x_1) + F_\sigma(x_1).
\]

As \( z + x \in T \) and \( z + \bar{x} \in T \), the first two members of the right part of the last equality are numerically \(< \epsilon/6 \), whence \( F_\sigma(\bar{x}) > 4\epsilon/6 \).

Note that \( \bar{x} \) is an arbitrary element of \( S(0, \alpha_1) \). Let \( \alpha_2 \) be the least non-negative residue of \( r \) mod \( \alpha_1 \), that is, \( r = k_1\alpha_1 + \alpha_2 \) \( (k_1 \) a positive integer, \( 0 \leq \alpha_2 < \alpha_1 \). Then \( \xi_0 \) can be represented in the form

\[
\xi_0 = u_1 + u_2 + \cdots + u_{k_1} + x_2,
\]

where \( u_i \cdot u_j = 0, \ (i \neq j) \), \( u_i x_2 = 0 \), \( \| u_i \| = \alpha_1 \), and \( \| x_2 \| = \alpha_2 \).

We have

\[
F_\sigma(x_2) = F_\sigma(\xi_0) - F_\sigma(u_1) - \cdots - F_\sigma(u_{k_1}).
\]

Since \( | F_\sigma(\xi_0) | < \epsilon/6 \) and \( F_\sigma(u_i) \geq 4\epsilon/6 \), we have

\[
F_\sigma(x_2) < \frac{\epsilon}{6} - k_1 \frac{4\epsilon}{6} \leq - \frac{3\epsilon}{6}.
\]
Repeating for $x_2$ and $S(0, \alpha_2)$ the reasoning given for $x_1$ and $S(0, \alpha_1)$, we find for $x \in S(0, \alpha_2)$, $F_v(x) < -\epsilon/6$.

Dividing $\alpha_1$ by $\alpha_2$, we obtain

$$\alpha_1 = k_2 \alpha_2 + \alpha_3, \quad (k_2 \text{ a non-negative integer, } 0 \leq \alpha_3 < \alpha_2).$$

If $x \in S(0, \alpha_3)$, then adding to it $k_3$ suitably selected elements $v_1, v_2, \cdots, v_{k_3}$ belonging to $S(0, \alpha_2)$, ($v_i \cdot v_j = 0$, when $i \neq j$, and $v_i \cdot x = 0$), we obtain an element $y \in S(0, \alpha_1)$. Then

$$F_v(x) = F_v(y) - F_v(v_1) - \cdots - F_v(v_{k_3}),$$

and

$$F_v(x) > \frac{4\epsilon}{6} + k_2 \frac{\epsilon}{6} \geq \frac{5\epsilon}{6}.$$

Divide $\alpha_2$ by $\alpha_3$, and so on. If $r$ and $\alpha_1$ are incommensurable, we shall obtain a sequence of spheres $S(0, \alpha_p)$ converging to zero, where

- on $S(0, \alpha_1)$, $F_v(x) > 4\epsilon/6$,
- on $S(0, \alpha_2)$, $F_v(x) < -\epsilon/6$,
- on $S(0, \alpha_3)$, $F_v(x) > 5\epsilon/6$,
- on $S(0, \alpha_4)$, $F_v(x) < -6\epsilon/6$,
- on $S(0, \alpha_5)$, $F_v(x) > 11\epsilon/6$,

and so on. Such a result contradicts the continuity of $F_v(x)$. If $r$ and $\alpha_1$ are commensurable, a certain $\alpha_M$ will be a divisor of $r$.

Then

$$\xi_0 = w_1 + w_2 + \cdots + w_\lambda,$$

where $w_i \cdot w_j = 0$, $(i \neq j)$, and $w_i \in S(0, \alpha_m)$, $(i = 1, 2, \cdots, \lambda)$. The numbers $F_v(w_i)$ have the same sign for every $i$ and $|F_v(w_i)| > \epsilon/6$. Consequently, $|F_v(\xi_0)| > \epsilon/6$, which is impossible, since $r > m_0$ and $\xi_0 \in H_{m_0}$. Thus the system (2') is really equally continuous.

Thus every sequence (2'), which satisfies condition (1') on $S(0, r)$ is equally continuous. Hence it follows that such a sequence converges to zero everywhere* on $R$, according to the general assertion.

* The reader sees that as far as it concerns the spherical surface $S(0, r)$ with the center at the point $x = 0$, the proof will not have to be changed considerably if we suppose that $H$, although not coinciding with $S(0, r)$, is of the second category in $S(0, r)$. 
3. Application of Functionals. Now let $S(x_0, r)$ be any spherical surface which does not pass through 0 and suppose that, for $x \in S(x_0, r)$, $\lim_{n} F_n(x) = 0$. Note that every spherical surface in $R$ has two centers and divides the whole space into two spheres, that is, that $S(x_0, r) = S(1-x_0, 1-r)$. Suppose that $x_0$ is that one of the two centers whose norm is greater than the radius.

For further reasoning it is necessary to show that $F_n(x_0) \to 0$. For that purpose consider the set of points $x \in S(x_0, r)$ for which $x \cdot x_0 = x$ and denote it by $S'$. If we introduce the set $J' = E_{x_0}(x(t) = 1)$, then $S' = S'(0, ||x|| - r)$ is a spherical surface with the center at 0 and with radius $||x_0|| - r$ in the space $R'$ of characteristic functions of measurable subsets of $J'$. Since reasoning in §2 remains valid when $J'$ is substituted for $J$, we have $F_n(x) \to 0$ everywhere in $R'$ and in particular at the point $x_0$.

Any element $x \in R$ can be represented in the form $x = x_0 - u(x) + v(x)$, where $u(x) = x_0(1-x)$ and $v(x) = x(1-x_0)$. Let us define a one-to-one and bicontinuous transformation of $R$ into itself according to the formulas:

$$y = \Psi(x) = u(x) + v(x) = x_0(1-x) + x(1-x_0),$$
$$x = \Psi^{-1}(y) = x_0 - yx_0 + y(1-x_0).$$

In this way $S(x_0, r)$ is transformed into $S(0, r)$.

Further define in the transformed space the sequence of functionals $\{ \Phi_n(y) \}$ on putting

$$\Phi_n(y) = F_n(x) - \frac{F_n(x_0)}{||x_0||} \int_{0}^{1} x_0(t)x(t)dt. \tag{A}$$

If we remember the definition of $F_n(x)$ and express $x$ in terms of $y$, we easily find that the functionals $\Phi_n(y)$ have the form required,

$$\Phi_n(y) = \int_{0}^{1} g_n(t)y(t)dt.$$

Since $F_n(x_0) \to 0$ and $F_n(x) \to 0$ on $S(x_0, r)$, we conclude from the equality (A) that $\Phi_n(y)$ converges to zero on $S(0, r)$. Applying the result of §2 to the sequence $\{ \Phi_n(y) \}$, we see that
\( \Phi_n(y) \rightarrow 0 \) everywhere in \( R \). Since the transformation is one-to-one, it follows from the formula (A) that \( F_n(x) \rightarrow 0 \) everywhere in \( R \).

Note that the statement made in §1 is not true for a spherical surface which passes through zero, \( S(x_0, \|x_0\|) \). This we can prove on putting

\[
F_n(x) = n \int_0^1 [x_0(t) - (1 - x_0(t))]x(t)dt.
\]

Let \( x \in S(x_0, \|x_0\|) ; x = x_0 - u + v \) and

\[
\|x_0\| = d(x, x_0) = \int_0^1 |x(t) - x_0(t)| dt = \int_0^1 | - u(t) + v(t)| dt = \int_0^1 [u(t) + v(t)] dt = \|u\| + \|v\|,
\]

that is, \( \|x_0 - u\| = \|v\| \). According to definition, \( x_0 - u = x_0 \cdot x \) and \( v = x(1 - x_0) \). Consequently for such an \( x \), \( F_n(x) = 0 \), for \( n = 1, 2, \ldots \). At the same time the sequence \( \{F_n(x)\} \) is by no means equally continuous in \( R \), as \( |F_n(x)| \rightarrow \infty \) for every \( x \) non-\( \in S(x_0, \|x_0\|) \).

4. Remark. In the paper of Saks mentioned above there are some vague points in the proof of Lemma 4. This refers to the way he motivates the inclusion (on page 553)

\[ H \subset H_1 + H_2 + \cdots. \]

The lemma itself is not true, at least for the space \( R \), as is shown by the example of the following sequence of transformations:

\[
\xi_n(x, t) = nx(t).
\]

Moreover, the assertions of Theorems 3 and 4 are not true in so far as they refer to \( R \), as is seen from the same example.

Institute of Mathematics,
Leningrad University