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so that, by Theorem 1 and its corollary,

\[ F_{89}(x) < F_{k_4}(x), \quad (x_8 < x < b); \]

in particular,

\[ F_{89}(x_7) < F_{k_4}(x_7). \quad (41) \]

Now (41) contradicts (39) and (40).

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SUFFICIENT CONDITIONS FOR A NON-REGULAR PROBLEM IN THE CALCULUS OF VARIATIONS*

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1. Introduction. Given \( J = \int_a^b f(x, y, y') \, dx \), it is well known that a minimizing curve satisfies the necessary conditions of Euler, Weierstrass, and Legendre, which we shall designate as I, II, and III,† respectively. If further, \( f_{y'y'}(x, y, y') \neq 0 \) on the minimizing curve, the Jacobi condition IV is necessary, while the stronger set of conditions I, II', III', and IV'‡ are sufficient for a strong relative minimum.

The purpose of this study is to obtain a set of sufficient conditions for a curve without corners along which \( f_{y'y'} \) may have zeros. Since the classical theory gives only the necessary conditions I, II, and III, we wish to obtain a Jacobi condition; and with this in view, introduce the integral

\[ L = \int_{x_1}^{x_2} \phi(x, y, y') \, dx, \quad \phi(x, y, y') = f(x, y, y') + k^2[y' - e'(x)]^2, \]

\( (x_1 \leq x \leq x_2, \ k \leq 0) \),

by means of which we find a necessary condition that we shall call IV′′\(_L\). Suitably strengthened, this becomes IV′′\(_L\), and the set of conditions I, II\(_b\), III\(_b\), and IV′′\(_L\), are found sufficient for an improper strong relative minimum.

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It appears likely that analogous results can be obtained for other problems in the Calculus of Variations and I hope to discuss some of these at a later time.

2. A Jacobi Necessary Condition. If $E: y = e(x)$ furnishes at least an improper strong relative minimum for $J$, it furnishes a proper strong relative minimum for $L$. Furthermore, if $E$ minimizes $J$ it satisfies III for $J$. This implies that it satisfies III' for $L$, since $\phi_{y'y'} = f_{y'y'} + 2k^2$; and the classical treatment then shows that it must satisfy IV for $L$.

If $E$ satisfies IV (or IV') for every $L$, we shall say that it satisfies the condition $IV_L$ (or $IV'_L$, respectively) for $J$. Clearly $IV_L$ is necessary. We now show that the same is true of $IV'_L$.

We write the parameter in $L$ in the form $k^2 = (a^2 + \alpha)/2$, $a \neq 0$, $\alpha > -a^2$, and consider the Jacobi differential equations*

\begin{align*}
(1) & \quad qu'' + ru' + su = 0, \\
(2) & \quad (q + a^2 + \alpha)u'' + ru' + su = 0,
\end{align*}

for $J$ and $L$, where $q = f_{y'y'}[x, e(x), e'(x)] \geq 0$ in the closed interval $[x_1, x_2]$ from III, and $r$ and $s$ are other known functions of $x$. Since $q$ may vanish in $[x_1, x_2]$, the usual existence theorems can not be applied to (1). They do apply to (2), however, the general solution of which for $\alpha = 0$ is $u = c_1u_1(x) + c_2u_2(x)$, where the $u$'s constitute a fundamental system and are of class $C''$ in $[x_1, x_2]$. $\Delta(x, x_1) = \pm u_2(x_1)u_1(x) \mp u_1(x_1)u_2(x)$ is a particular solution vanishing at $x = x_1$. By hypothesis, $E: y = e(x)$ is a minimizing curve satisfying $IV_L$ so that, by proper choice of signs, $\Delta(x, x_1)$ is positive in the interval $x_1 < x < x_2$.

For every admissible $\alpha$ (that is, $\alpha > -a^2$) there exists a solution $\Delta(x, x_1, \alpha)$ of (2) vanishing at $x = x_1$ and such that $\Delta'(x_1, x_1, \alpha) = \Delta'(x_1, x_1)$, where $\Delta''(x, x_1, \alpha)$ is continuous in $x$ and of class $C'$ in $\alpha$.†

We next study the related equation

\begin{equation}
(q + a^2)u'' + ru' + su = -\alpha\Delta''(x, x_1, \alpha),
\end{equation}

* Oskar Bolza, Vorlesungen über Variationsrechnung, 1933, p. 60.
† That is, they have continuous second derivatives. Bolza, loc. cit., p. 14.
‡ Replace (2) by the system $u' = v$ and $(q + a^2 + \alpha)v' + rv + su = 0$, and apply the existence theorem given by Bolza, loc. cit., p. 187.
whose general solution can, by the method of variation of parameters, be expressed in the form

\[ u = c_1 u_1(x) + c_2 u_2(x) + \alpha A(x, \alpha), \]

where

\[ A(x, \alpha) = u_1(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha)u_2(x)dx}{(q + a^2)D(x)} - u_2(x) \int_{x_1}^{x} \frac{\Delta''(x, x_1, \alpha)u_1(x)dx}{(q + a^2)D(x)}, \]

\[ D(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \neq 0 \text{ in the closed interval } [x_1, x_2]. \]

\[ \Delta(x, x_1, \alpha), \text{ as a particular solution of (3), can be represented in the form (4); and, since it vanishes for } x = x_1, \text{ we obtain} \]

\[ (5) \quad \Delta(x, x_1, \alpha) = \lambda \Delta(x, x_1) + \alpha A(x, \alpha), \]

where in general \( \lambda \) is a function of \( \alpha \). Clearly \( \lambda(0) = 1 \).

\( E \) satisfies IV by hypothesis. If it fails to satisfy IV' for the \( L \) corresponding to \( \alpha = 0 \), we have

\[ \Delta(x_2, x_1, 0) = \lambda(0)\Delta(x_2, x_1) = \Delta(x_2, x_1) = 0, \]

while, if a second \( \alpha \neq 0 \) has the same property, we have

\[ \Delta(x_2, x_1, \alpha) = \lambda(\alpha)\Delta(x_2, x_1) + \alpha A(x_2, \alpha) = \alpha A(x_2, \alpha) = 0. \]

This requires

\[ (6) \quad A(x_2, \alpha) = 0. \]

But

\[ A(x_2, \alpha) = \int_{x_1}^{x_2} \frac{\Delta''(x, x_1, \alpha) \Delta(x, x_2)}{(q + a^2)D(x)} dx \]

\[ = \Delta(x_2, x_2) \int_{x_1}^{x_2} \frac{\Delta''(x, x_1, \alpha) dx}{(q + a^2)D(x)} \]

\[ = \Delta(x_2, x_2) [\Delta'(x_1, x_1, \alpha) - \Delta'(x_2, x_1, \alpha)] \]

\[ (q + a^2)D(x) \]

\[ (x_1 < x_2; \bar{q} = q(x)), \]

* Bolza, loc. cit., p. 75.
where $\Delta(x, x_2)$ is written for $u_2(x_2)u_1(x) - u_1(x_2)u_2(x)$. This fraction cannot vanish, the first factor in the numerator being different from zero by IV$_L$, the second factor being the difference between two terms of opposite sign. Thus (6) is false; and there is at most one $L$, namely the one for which $\alpha=0$, for which $E$ fails to satisfy IV$'$. 

If $\Delta(x_2, x_1, 0) = 0$, we have $\Delta(x_2, x_1, \alpha) = \alpha A(x_2, \alpha)$ from (5). Furthermore $\Delta(x_2, x_1, \alpha)$ must then have a minimum of zero for $\alpha=0$;* so that its derivative, which is $A(x_2, 0)$, must vanish. This is a special case of (6), which has been proved to be false, so that IV$_L$ is a necessary condition.

3. *Sufficient Conditions for a Minimum for $L$. We assume an arc $E: y = e(x)$ satisfying the necessary conditions I, II, III, and IV$_L$ for $J$. If II is strengthened to II$_8$, we can show that this arc satisfies the classical sufficient conditions for $L$.

Comparing the Euler equations, we see that if $E$ satisfies I for $J$ it does the same for $L$. The $E$-functions† for the two problems are related by the equation

$$E_L(x, y, y', Y') = E_J(x, y, y', Y') + k^2(y' - Y')^2,$$

so that II$_8$ for $J$ implies II$_8'$ for $L$. We have seen in §2 that III for $J$ implies III$'$ for $L$ and the condition IV$_L$ requires IV$'$ for $L$ as a matter of definition.

Hence $E$ furnishes a proper strong minimum to $L$ relative to a certain $(x, y)$ region $R$, which in general depends on $k$.

4. *Sufficient Conditions for an Improper Strong Relative Minimum for $J$. We must find how to strengthen our conditions so as to insure a field§ $F$ which is independent of $k$. To that end we replace III by III$_8$ and consider the line $A: x = x_1, y = n\lambda - y_1$, together with a slope function $P(\lambda) = m\lambda + e'(x_1)$. The extremals for $L$ are $y = y(x, a, b, \alpha)$, and the equations

* $\Delta(x_2, x_1, \alpha) > 0$ for $\alpha \neq 0$ by IV$_L$ and the choice of signs preceding equation (3).
† This is the only direct reference to the $E$-function. There need be no confusion with our notation for the curve $E: y = e(x)$.
‡ If $E$ satisfies III$'$ and IV, but not IV$'$, for $J$, $R$ reduces to the curve $E$ as $k$ approaches zero.
§ Bliss, loc. cit., pp. 132–33.
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\[ n\lambda + y_1 - y(x_1, a, b, \alpha) = 0, \]
\[ m\lambda + e'(x_1) - y'(x_1, a, b, \alpha) = 0, \]
define \( a = a(\lambda, \alpha) = \delta(y, \alpha) \), and \( b = b(\lambda, \alpha) = \delta(y, \alpha) \) for any admissible \( \alpha \) and for every \( y \) for which \((x_1, y)\) is in the region where \( \text{III}_b \) holds. These implicit functions are of at least class \( C' \) in their respective variables. We thus have a family of extremals of parameter \( \lambda \) for each admissible \( \alpha \),

\[ y = \phi(x, \lambda, \alpha) = y[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha], \]

intersecting \( \Lambda \) and including \( E \) for \( \lambda = 0 \). We wish this family to furnish a field.

If there exists an \( x, (x_1 < x \leq x_2) \), such that \( \phi(x, \lambda_1, \alpha) - \phi(x, \lambda_2, \alpha) = 0 \), there is a \( \bar{\lambda}, (\lambda_1 < \bar{\lambda} < \lambda_2) \), such that

\[ \phi_\lambda(x, \bar{\lambda}, \alpha) = y_a \frac{\partial a}{\partial \lambda} + y_b \frac{\partial b}{\partial \lambda} \bigg|_{\lambda = \bar{\lambda}} = 0. \]

This can be expressed in the form\( \dagger \)

\[ \begin{vmatrix} n & y_a(x) & y_b(x) \\ m & y_a(x) & y_b(x) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} \]

\[ D_1 = \begin{vmatrix} y_a(x_1) & y_b(x_1) \\ y_a'(x_1) & y_b'(x_1) \end{vmatrix} \neq 0. \] \( \ddagger \)

We shall say that \( E \) satisfies the condition \( \text{IV}_b' \) if constants \( \delta > 0, \eta > 0, \) and \( A \) exist such that\( \S \)

\[ \Delta(x, x_1, y, \alpha) = \begin{vmatrix} \tilde{y}_a(x) & \tilde{y}_b(x) \\ \tilde{y}_a(x_1) & \tilde{y}_b(x_1) \end{vmatrix} \]

is, in absolute value, greater than \( \delta \) in the region \( x_1 < x \leq x_2, |y - y_1| \leq \eta, A \geq \alpha > -a^2 \). The first determinant in (7) has a finite limit as \( n \) approaches zero; and hence, if \( n \) is small in absolute value, \( \text{IV}_b' \) insures that the expression will not vanish and that no two extremals of the family pass through the same

\* \( a, b, \delta, \) and \( \delta \) also depend on \( m \) and \( n \), which are omitted in the notation.
\( \dagger \) \( y_a(x), \cdots \) are written for \( y_a[x, a(\lambda, \alpha), b(\lambda, \alpha), \alpha], \cdots \).
\( \ddagger \) The method used by Bolza, loc. cit., pp. 73–75, shows that \( D_1 \neq 0. \)
\( \S \) \( \tilde{y}_a(x), \cdots \) are written for \( y_a[x, \delta(y, \alpha), \delta(y, \alpha)\alpha], \cdots \).
point. This condition also requires $\phi$ to be strictly monotone in $\lambda$ for a given $x$ and $\alpha$, so that an extremal of the family passes through each point of a certain region $F$ about $E$. The region $F$ is a field and is independent of $k$ (that is, of $\alpha$).*

Finally, if $E$ satisfies I, II, III, and IV$_{Lb}$, we have $L(E) < L(C)$ for every $C \neq E$ in $F$. But

$$L(C) = J(C) + \epsilon, \quad \epsilon > 0, \quad \lim_{k \to 0} \epsilon = 0.$$ 

Furthermore $L(E) = J(E)$, so that $J(E) < J(C) + \epsilon$, and finally $J(E) \leq J(C)$.

5. Applications. The line $y = 0$ is an extremal for a problem involving any one of the following integrands:

$$f \equiv M(x, y) + N(x, y)y', \quad M_y = N_x,$$

$$f \equiv x^2 + y^2 + yy', \dagger$$

$$f \equiv y^4.$$

Our sufficient conditions for an improper minimum are met by $y = 0$ in each case, but III' is not met for any of them.

* Condition IV$_{Lb}$ could be replaced by the following. There exist constants $\eta > 0$, $\xi > 0$, and $A$ such that $E$ satisfies III for $x_0 \leq x \leq x_2$, $x_0 = x_1 - \xi$ and such that $A(x, x_0, y, \alpha) \neq 0$ for $x_0 = x_1 - \xi$, $x_0 < x \leq x_2$, $|y - y_1| \leq \eta$, $A \geq \alpha > -a^2$. See Bolza, loc. cit., bottom p. 103.

† An example given by Bolza, loc. cit., p. 35.