ON AN INTEGRAL TEST OF R. W. BRINK FOR THE CONVERGENCE OF SERIES

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1. Introduction. The test in question is embodied in the following theorem due to R. W. Brink.*

Let \( \sum^\infty u_n \) be a series of positive terms. Also let \( r(x) \) be a function such that (i) \( r(n) = n = u_n + 1/u_n \), (ii) \( 0 < \lambda \leq r(x) \leq \mu \), (iii) \( r'(x) \) exists and is continuous, \( \int^\infty |r'(x)| \, dx \) is convergent. Then the convergence of the integral

\[
\int^\infty e^{\int^x \log r(t) \, dt} \, dx
\]

is necessary and sufficient for the convergence of the series \( \sum^\infty u_n \).

It is the object of this note to show that Brink's theorem can be expressed in a more general form (Theorem 3 below) which leads at once to all the ratio tests for the convergence of series associated with Kummer's test. The ratio tests are thus welded into unity from a point of view somewhat different from that adopted by Pringsheim in his classical paper Allgemeine Theorie der Divergenz und Convergenz von Reihen mit positiven Gliedern.†

2. Connection of Brink's Theorem with the Maclaurin-Cauchy Integral Test. The problem which confronts us in Brink's theorem is clearly that of setting up an integral \( \int^\infty F(t) \, dt \) whose behaviour at infinity is reflected by a given series \( \sum^\infty u_n \). When \( \sum^\infty u_n \) has all but a finite number of terms positive, the method employed to establish the Maclaurin-Cauchy integral test shows that the convergence of \( \int^\infty F(x) \, dx \) is sufficient for that of \( \sum^\infty u_n \) if for \( n \leq x \leq n + 1 \), \( 0 < u_n \leq F(x) \), \( (n = m, m + 1, \cdots) \). Denoting \( u_{n+1}/u_n \) by \( r_n \), the condition assumed is that

\[
r_{n-1} \cdot r_{n-2} \cdots r_m \leq \frac{F(x)}{u_m}, \quad (n \leq x \leq n + 1),
\]

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* R. W. Brink, A new integral test for the convergence and divergence of infinite series, Transactions of this Society, vol. 19 (1918), p. 188.
that is,
\[
\sum_{r=m}^{n-1} \log r_r \leq \log \frac{F(x)}{F(m)} + \log \frac{F(m)}{u_m} = \int_m^x \frac{F'(t)}{F(t)} \, dt + \log \frac{F(m)}{u_m},
\]
or,
\[
\sum_{r=m}^{n-1} \left[ \log r_r - \int_r^{r+1} \frac{F'(t)}{F(t)} \, dt \right] - \int_n^x \frac{F'(t)}{F(t)} \, dt \leq \log \frac{F(m)}{u_m},
\]
\(n \leq x \leq n + 1\).

The right-hand member of the above inequality may be altered to any arbitrary constant; for this would merely imply the multiplication of \(F(x)\) by a positive constant in our initial hypothesis. Also, for the truth of the altered inequality the following conditions are sufficient:

(i) \(F'(x)/F(x)\) is bounded and integrable for \(x \geq m\),

(ii) \(\log r_r - \int_r^{r+1} \frac{F'(x)}{F(x)} \, dx \leq \delta_r\),

where \(\sum \delta_r\) is bounded above as \(n \to \infty\), which is a consequence of

\[\log r_r - \frac{F'(x)}{F(x)} \leq \delta_r, \quad (\nu \leq x \leq \nu + 1).\]

If we put \(F'(x)/F(x)=f(x)\), the integral whose convergence is sufficient for that of \(\sum u_n\) assumes the form \(\int e^{-\int f(y) \, dy} \, dx\). Further, the divergence of this integral is sufficient for the divergence of \(\sum u_n\) provided that in (ii) above the inequality sign is reversed and \(\sum \delta_r\) is bounded below. Hence we are led to formulate the test as follows.

**Theorem 1.** Let \(\sum u_n\) be a series of positive terms and \(r_n = u_{n+1}/u_n\). If

(i) \(f(x)\) is bounded and integrable for \(x \geq m\), and
\[ \int_{0}^{\infty} e^{\int_{x}^{\infty} f(t) \, dt} \, dx \text{ is convergent,} \]

\{ or \ (D): \quad \int_{0}^{\infty} e^{\int_{x}^{\infty} f(t) \, dt} \, dx \text{ is divergent} \};

(ii) \text{ for } n \leq x \leq n+1,

\[ \log r_{n} \leq f(x) + \delta_{n}, \sum_{n} \delta_{n} \text{ being bounded above}, \]

\{ or \ (D): \quad \log r_{n} \geq f(x) + \delta'_{n}, \sum_{n} \delta'_{n} \text{ being bounded below} \};

\text{then } \sum_{n} u_{n} \text{ is convergent } \{ \text{or divergent} \}.

The direct proof of the theorem is exactly on the lines of that of Theorem 2 given below.

**Brink's Integral Test.** This is an immediate deduction from Theorem 1. For if \( r(x) \) is defined as in Brink's theorem, then

\[ \log r(x) - \log r_{n} = \int_{n}^{x} \frac{r'(t)}{r(t)} \, dt, \]

and

\[ | \log r(x) - \log r_{n} | \leq \frac{1}{\lambda} \int_{n}^{n+1} | r'(t) | \, dt, \quad (n \leq x \leq n+1). \]

Hence replacing \( f(x) \) by \( \log r(x) \) and taking

\[ \delta_{n} = \frac{1}{\lambda} \int_{n}^{n+1} | r'(t) | \, dt, \quad \delta'_{n} = -\frac{1}{\lambda} \int_{n}^{n+1} | r'(t) | \, dt, \]

we see that \( \sum_{n} u_{n} \) converges or diverges with

\[ \int_{0}^{\infty} e^{\int_{x}^{\infty} \log r(t) \, dt} \, dx. \]

Thus Theorem 1 includes Brink's integral test, as one of his own theorems in the Annals of Mathematics* includes Hardy's generalization of the Maclaurin-Cauchy integral test†


3. Preliminary Theorems and Deductions. Theorem 1 admits of the following generalization.

**Theorem 2.** Let $\sum^\infty u_n$ be a series of positive terms and $r_n = u_{n+1}/u_n$. If

(i) $(D_n)$ is a strictly increasing sequence tending to infinity;

(ii) $d_n = D_n - D_{n-1} = O(1)$;

(iii) $f(x)$ is bounded and integrable for $x \geq D_m$, and

1. $\int_0^\infty e^{\int^t f(s)ds} dt \, dx$ is convergent,

or

2. $\int_0^\infty e^{\int^t f(s)ds} dt \, dx$ is divergent;

(iv) for $D_{n-1} \leq x \leq D_n$,

1. $\frac{1}{d_n} \log r_n \leq f(x) + \delta_n$, $\sum^n \delta_d d_v$ being bounded above,

or

2. $\frac{1}{d_n} \log r_n \geq f(x) + \delta'_n$, $\sum^n \delta'_d d_v$ being bounded below

then $\sum^\infty u_n d_n$ is convergent {or divergent}.

**Proof of Case (C).** Since

$$\frac{1}{d_v} \log r_v \leq f(t) + \delta_v,$$

by integration,

$$\log r_v \leq \int_{D_{v-1}}^{D_v} f(t) dt + \delta_d v.$$

Sum for $v = m+1, m+2, \ldots, n-1$; then

$$\log \frac{u_n}{u_{m+1}} \leq \int_{D_m}^{D_{n-1}} f(t) dt + \sum^{n-1}_{r=m+1} \delta_d v$$

$$< \int_{D_m}^{D_{n-1}} f(t) dt + K_1, \quad (K_1 \text{ fixed})$$

Also, for $D_{n-1} \leq x \leq D_n$, since $|f(t)| < M$ (fixed), $d_n < K$ (fixed), we have
- $KM < \int_{D_{n-1}}^{x} f(t) dt$.

Add the last two inequalities; then

$$\log \frac{u_n}{u_{m+1}} < \int_{D_m}^{x} f(t) dt + K_1 + KM, \quad (D_{n-1} \leq x \leq D_n),$$

and

$$u_n < u_{m+1} e^{K_1 + KM} \int_{D_{n-1}}^{D_n} f(t) dt, \quad (D_{n-1} \leq x \leq D_n).$$

Hence, integrating from $D_{n-1}$ to $D_n$, we have

$$u_n d_n < u_{m+1} e^{K_1 + KM} \int_{D_{n-1}}^{D_n} f(t) dt.$$

Compare the series $\sum_{n=0}^{\infty} u_n d_n$ with the series of positive terms $\sum_{n=0}^{\infty} e^{K_1 + KM} \int_{D_{n-1}}^{D_n} f(t) dt dx$, and the test for convergence ($Q$) follows at once. The test for divergence ($D$) is similarly proved.

The following is an adjunct to Theorem 2.

**Theorem 2a.** In Theorem 2, suppose the condition $d_n = O(1)$ is dropped and $f(x) < 0$ (that is, the integrand in the test integral is a strictly decreasing function). If other conditions remain the same, $\sum_{n=0}^{\infty} u_n d_n$ is convergent in case ($Q$) and $\sum_{n=0}^{\infty} u_n d_n$ is divergent in case ($D$).

A slight modification is required in our former proof of ($Q$):

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{D_n} f(t) dt + K_1.$$

Also, for $D_{n-1} \leq x \leq D_n$,

$$0 \leq - \int_{x}^{D_n} f(t) dt.$$

We add the last two inequalities, and obtain

$$\log \frac{u_{n+1}}{u_{m+1}} < \int_{D_m}^{x} f(t) dt + K_1, \quad (D_{n-1} \leq x \leq D_n);$$

whence, as before,
\[ u_{n+1}d_n < u_{m+1}e^{\int_{D_{n-1}}^{D_n} e^{-\int_{D_{n-1}}^{D_n} f(t)dt} dx}, \]

and the desired result follows by comparison.

**Deductions from Theorem 2a.** (i) Taking \( f(x) = -\rho < 0 \), \( \delta_n = 0 \), and putting \( u_{n+1}d_n = a_n \), we see that the condition

\[
\frac{1}{d_n} \log \frac{a_n \cdot d_{n-1}}{a_{n-1} \cdot d_n} \leq -\rho < 0
\]

is sufficient* for the convergence of \( \sum a_n \).

(ii) The condition

\[
\frac{1}{d_n} \log \frac{a_n \cdot d_{n-1}}{a_{n-1} \cdot d_n} \leq -\frac{1}{D_{n-1}} - \frac{1}{D_{n-1} \cdot l_1D_{n-1}} - \cdots - \frac{\alpha}{D_{n-1} \cdot l_1D_{n-1} \cdots l_pD_{n-1}},
\]

\((\alpha > 1),\)

where \( l_1D_{n-1} = \log D_{n-1}, l_2D_{n-1} = \log \log D_{n-1}, \cdots \) (and \( n \geq m + 1 \) which is such that \( l_pD_m > 0 \)), is sufficient for the convergence of \( \sum a_n \). For this implies that

\[
\frac{1}{d_n} \log \frac{u_{n+1}}{u_n} \leq -\frac{1}{x} - \frac{1}{x \cdot l_1x} - \cdots - \frac{\alpha}{x \cdot l_1x \cdots l_px},
\]

\((\alpha > 1; D_{n-1} \leq x \leq D_n).\)

Hence taking

\[ f(x) = -\frac{1}{x} - \frac{1}{x \cdot l_1x} - \cdots - \frac{\alpha}{x \cdot l_1x \cdots l_px}, \quad (\alpha > 1); \delta_n = 0, \]

we deduce the convergence of \( \sum a_n \).

(iii) Similarly the condition

\[
\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \geq -\frac{1}{D_n} - \frac{1}{D_n \cdot l_1D_n} - \cdots - \frac{\alpha}{D_n \cdot l_1D_n \cdots l_pD_n},
\]

\((\alpha \leq 1),\)

is sufficient for the divergence of \( \sum a_n \).

Setting \( D_n = n \) in (ii) and (iii), we obtain Bertrand's logarithmic criteria for convergence and divergence.

* A. Pringsheim, loc. cit., p. 370.
4. Generalization of Brink’s Theorem.

**Theorem 3.** Let $\sum a_n$ be a series of positive terms. If

(i) $(D_n)$ is a strictly increasing sequence tending to infinity;

(ii) $d_n = D_n - D_{n-1} = O(1)$;

(iii) $f(x)$ has a continuous derivative $f'(x)$ and $\int f'(x) dx$ is convergent;

(iv) 

\[
\int_{C} e^{f(x)} dx \text{ is convergent,}
\]

\{or, (D):

\[
\int_{D} e^{f(x)} dx \text{ is divergent};
\]

(v)

\[
\frac{1}{d_n} \log \frac{a_{n+1}}{a_n} d_n \leq f(D_n),
\]

\{or, (D):

\[
\frac{1}{d_n} \log \frac{a_{n+1}}{a_n} d_n \geq f(D_n)\}

\]

then $\sum a_n$ is convergent \{or divergent\}.

**Proof of (C).** Denoting $a_n/d_n$ by $u_n$, we have in the notation of Theorem 2,

\[
\frac{1}{d_n} \log r_n \leq \int_{x}^{D_n} f'(t) dt + f(x)
\]

\[
\leq \int_{D_{n-1}}^{D_n} |f'(t)| dt + f(x), \quad (D_{n-1} \leq x \leq D_n).
\]

Whence, choosing $\varepsilon = \int_{D_{n-1}}^{D_n} |f'(t)| dt$ in Theorem 2, we deduce the convergence of $\sum u_n d_n = \sum a_n$.

Proof of (D) is similar.

**Deductions from Theorem 3.** (i) If

\[
\frac{1}{d_n} \log \frac{a_{n+1}}{a_n} d_n = f(D_n),
\]

then, under the conditions assumed, the convergence of

\[
\int e^{f(x)} dx
\]
is necessary and sufficient for the convergence of $\sum^\infty a_n$. When $D_n = n$, we have Brink's theorem.

(ii) Taking $f(x) = -\rho < 0$, we see that the condition

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \leq -\rho < 0$$

is sufficient for the convergence of $\sum^\infty a_n$. Since $\log \gamma \leq \gamma - 1$, ($\gamma > 0$), it follows that the above condition can also be expressed in Kummer's form:

$$\frac{1}{d_n} \left( \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1 \right) \leq -\rho < 0.$$

(iii) Taking

$$f(x) = -\frac{1}{x} - \frac{1}{x \cdot l_1 x} - \cdots - \frac{\alpha}{x \cdot l_1 x \cdots l_p x}, \ (\alpha > 1),$$

we observe that the condition

$$\frac{1}{d_n} \log \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} \quad \text{and} \quad \frac{1}{d_n} \left( \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1 \right) \quad \text{or}, \quad \frac{1}{d_n} \left( \frac{a_{n+1} \cdot d_n}{a_n \cdot d_{n+1}} - 1 \right) \leq \frac{1}{D_n} - \frac{1}{D_n \cdot l_1 D_n} - \cdots - \frac{\alpha}{D_n \cdot l_1 D_n \cdots l_p D_n}, \ (\alpha > 1),$$

is sufficient for the convergence of $\sum^\infty a_n$.

The corresponding divergence criterion has already been given.

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* A. Pringsheim, loc. cit., p. 371.
† A. Pringsheim, loc. cit., p. 361, footnote.