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ON SOME GAP THEOREMS FOR EULER’S METHOD OF SUMMATION OF SERIES

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Hardy and Littlewood* have proved the following theorem:

For a given series \( \sum_{k=0}^{\infty} a_{nk} \), \( a_{nk} \neq 0 \), let \( \theta \) be a fixed constant such that

\[
\frac{n_{k+1}}{n_k} \geq \theta > 1, \quad (k = 1, 2, \cdots).
\]

If this series be summable by Abel’s method of summation to the sum \( s \), then this series is convergent and its sum is \( s \).

Obreschkoff† obtained also a similar result for Cesàro’s method. We shall now study these results for Euler’s method.

We shall begin with the following theorem:

**Theorem 1.** Let \( \sum_{n=0}^{\infty} a_n \) be a given series summable by Euler’s method, that is, if \( s_0 = 0 \), \( s_n = a_0 + a_1 + \cdots + a_{n-1} \), \( n \geq 1 \),

\[
\lim_{n \to \infty} \frac{1}{2^n} \left\{ s_0 + ns_1 + \frac{n(n-1)}{2!} s_2 + \cdots + s_n \right\} = s
\]

exists; and for two given increasing sequences \( \{n_k\}, \{n'_k\} \), \( n_k < n'_k \), of integers and for a given number \( \alpha \), \( 1 \leq \alpha < 2 \), let

\[
a_v = 0, \text{ for } n_k < n < n'_k, \quad (k = 1, 2, \cdots),
\]

\[
a_n = O(\alpha^n).
\]

If \( \eta_k / \eta_k \geq (1 + \eta)/(1 - \eta), \quad (k = 1, 2, \cdots) \), for a positive number \( \eta \) such that

\[
(1 + \eta) \log (1 + \eta) + (1 - \eta) \log (1 - \eta) - 2 \log \alpha > 0,
\]

then

\[
\lim_{k \to \infty} \sum_{v=0}^{n_k} a_v = s.
\]

‡ If, in this theorem, (1) holds uniformly and \( O \) of (2) is independent of \( z \) when each \( a_n \) is a function of \( z \), then (3) also holds uniformly.
PROOF. To prove this, we can consider that all $n_k' - n_k - 1$ are even. Then putting

$$n_k + \frac{n_k' - n_k - 1}{2} + 1 = m$$

we have

$$a_{m-1} = a_{m-2} = \ldots = a_{n_k+1} = 0,$$

$$a_m = a_{m+1} = \ldots = a_{n_k' - 1} = 0.$$  

Hence, if we put

$$s_n' = \frac{1}{2^n} \left\{ s_0 + n s_1 + \frac{n(n - 1)}{2!} s_2 + \ldots + s_n \right\},$$

then we have

$$s_{2m} - s_m = \frac{1}{2^{2m}} \left\{ s_0 + 2m s_1 + \frac{2m(2m - 1)}{2!} s_2 + \ldots + s_{2m} \right\}
- \frac{1}{2^{2m}} \left\{ s_m + 2ms_m + \frac{2m(2m - 1)}{2!} s_m + \ldots + s_m \right\}
= \frac{1}{2^{2m}} \left\{ (a_0 + \ldots + a_{n_k}) - 2m(a_1 + \ldots + a_{n_k})
- \frac{2m(2m - 1)}{2!} (a_2 + \ldots + a_{n_k}) - \ldots
- \frac{2m(2m - 1) \cdots (2m - n_k + 1)}{n_k!} a_{n_k}
+ \frac{2m(2m - 1) \cdots (2m - n_k')}{(n_k' + 1)!} a_{n_k} + \ldots
+ (a_{n_k} + \ldots + a_{2m-1}) \right\}.$$  

Since from (2) we can find a positive constant $M$ such that

$$|a_n| < Me^{\alpha n}, \quad (n = 0, 1, 2, \ldots),$$

we get

$$|s_{2m} - s_m| < 2M \frac{e^{2m \log \alpha}}{2^{2m}} (n_k + 1)^2 \frac{2m(2m - 1) \cdots (2m - n_k + 1)}{n_k!}
< 4M \frac{e^{2m \log \alpha}}{2^{2m}} \frac{2m \Gamma(2m)}{\Gamma(m + \lambda) \Gamma(m - \lambda)},$$

where $\lambda = m - n_k$.  

Let us now put
\[ f(m) = \frac{e^{2m\log a}}{2^m} \frac{2m\Gamma(2m)}{\Gamma(m - \lambda)\Gamma(m + \lambda)}, \]
or
\[ f(m) = \frac{e^{2m\log a}}{2^m} \frac{2m\Gamma(2m)}{\Gamma(m(1 - \delta))\Gamma(m(1 + \delta))}, \]
where \( \lambda = m\delta, \) \((0 < \delta = (n'_k - n_k + 1)/(n'_k + n_k + 1) < 1).\) Then
\[
\log f(m) = 2m \log \alpha - 2m \log 2 + \log (2m) \\
+ (2m - \frac{1}{2}) \log (2m) - 2m + O(1) \\
- \left\{ m(1 - \delta) - \frac{1}{2} \right\} \log ((1 - \delta)m) + (1 - \delta)m + O(1) \\
- \left\{ m(1 + \delta) - \frac{1}{2} \right\} \log ((1 + \delta)m) + (1 + \delta)m + O(1) \\
= -m\phi(\delta) + \frac{1}{2} \log m + O(1),
\]
where
\[
\phi(\delta) = (1 + \delta) \log (1 + \delta) + (1 - \delta) \log (1 - \delta) - 2 \log \alpha.
\]
For a fixed number \( \eta_0, \) \((1 > \eta_0 \geq 0),\) such that \( \phi(\eta_0) = 0,\) any number \( \eta \) such that \( 1 > \eta > \eta_0 \) gives \( \phi(\eta) > 0.\) When \( \eta \) is so fixed, it follows from (1) that
\[
\lim s_m = \lim s_{2m} = s \quad \text{for} \quad 1 > \delta > \eta.
\]
On the other hand, from \( n'_k / n_k \geq (1 + \eta)/(1 - \eta) \) we have
\[
\frac{n'_k - n_k + 1}{n'_k + n_k + 1} > \eta.
\]
Consequently \( 1 > \delta > \eta \) since
\[
\delta = (n'_k - n_k + 1)/(n'_k + n_k + 1).
\]
Therefore
\[
\lim_{k \to \infty} s_{n_k} = \lim_{m \to \infty} s_m = s.
\]
Thus our theorem is completely proved.
REMARK. From Theorem I and Knopp's theorem follows immediately Ostrowski's theorem:

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be a power series whose radius of convergence is 1. If

\[ a_r = 0 \quad \text{for} \quad n_k < r < n'_k, \]

and

\[ \frac{n'_k}{n_k} > 1 + \theta, \quad (k = 1, 2, \cdots), \]

\( \theta \) being a positive constant, then the partial sums \( s_{nk} \) of this series converge uniformly in a full neighbourhood of every regular point of the function (z) on the unit circle.

THEOREM 2. Let \( \sum a_n \) be a given series summable by Euler's method to the sum \( s \), and for two given increasing sequences \( \{ n_k \} \), \( \{ n'_k \} \), \( (n_k < n'_k) \), of positive integers and for \( p \geq -1 \) let

\[ a_r = 0 \quad \text{for} \quad n_k < r < n'_k, \quad (k = 1, 2, \cdots), \]

\[ a_n = O(n^p). \]

If

\[ \frac{n'_k}{n_k} \geq 1 + c \left( \frac{3}{2} + p \right)^{1/2} \left( \frac{\log n_k}{n_k} \right)^{1/2} \]

holds for any given number \( c \) greater than 2, and for all sufficiently great integers \( k \), then we have

\[ \lim_{k \to \infty} \sum_{r=0}^{n_k} a_r = s. \]

PROOF. From the assumption we can take \( c' \) and \( q \) such that

\[ \frac{n'_k - n_k}{n_k} \geq c' \left( \frac{3}{2} + p + q \right)^{1/2} \left( \frac{\log n_k}{n_k} \right)^{1/2}, \quad (c' > 2, q > 0). \]

Therefore

\[
\frac{n'_k - n_k}{n'_k + n_k} \geq \left( \frac{3}{2} + \rho + q \right)^{1/2} \left( \frac{\log n_k}{n'_k + n_k} \right)^{1/2} \\
\geq c \left( \frac{3}{2} + \rho + q \right)^{1/2} \left( \log n_k + 2 \right) \\
\geq \left( \frac{3}{2} + \rho + q \right)^{1/2} \left( \log n_k \right)^{1/2}
\]

for all sufficiently great integers \( k \). Hence from

\[
\delta = \frac{n'_k - n_k + 1}{n'_k + n_k + 1} > \frac{n'_k - n_k}{n'_k + n_k}
\]

we get

\[
\delta^2 \geq \left( \frac{3}{2} + \rho + q \right) \frac{\log m}{m}, \quad \left( m = \frac{n'_k + n_k + 1}{2} \right).
\]

Consequently, from

\[
(1 + \delta) \log (1 + \delta) + (1 - \delta) \log (1 - \delta) > \delta^2,
\]

we have

\[
(1 + \delta) \log (1 + \delta) + (1 - \delta) \log (1 - \delta) > \left( \frac{3}{2} + \rho + q \right) \frac{\log m}{m},
\]

whence as in the proof of Theorem I we obtain

\[
\left| s'_m - s_m \right| < M \frac{(2m)^{\nu}}{2^{2m} \Gamma(2m) \Gamma(m(1 - \delta)) \Gamma(m(1 + \delta))} \\
< M' \exp \left\{ m \left\{ \left( \frac{3}{2} + \rho \right) \frac{\log m}{m} - (1 + \delta) \log (1 + \delta) \\
- (1 - \delta) \log (1 - \delta) \right\} + O(1) \right\} \\
\rightarrow 0, \quad (m \rightarrow \infty),
\]

\( M, M' \) being constants. Therefore

\[
\lim s'_m = \lim s_m = s.
\]

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