\[
\left| \int_{a}^{b} d[f(x)g(x)] - \int_{a}^{b} g(x)df(x) - \int_{a}^{b} f(x)dg(x) \right|
\]

is less than \( \epsilon \) multiplied by the total variation of \( f(x) \) plus the total variation of \( g(x) \). Hence this expression must be equal to 0 and therefore the formula for integration by parts is valid under the above hypotheses.

**University of Michigan**

**A SET OF POSTULATES FOR BOOLEAN ALGEBRA**

**BY SOLOMON HOBERMAN AND J. C. C. MCKINSEY**

1. A *New Set of Postulates.* In the development of a Boolean Algebra, Boole's Law of Development

   \[ f(x) = f(1)x + f(0)x', \]

stands out as a basic relationship. This law is so all embracing that the question naturally arises, if this is set as a postulate, what postulates in addition to it are needed to define a Boolean Algebra? Using as undefined a class \( K \) and the Sheffer stroke function, we shall show that, in addition to a form of Boole's Law, only two "trivial" postulates are required.

**Postulates.*

I. \( K \) contains at least two elements.

II. If \( a \) and \( b \) are elements of \( K \), then \( a/b \) is an element of \( K \).

Definitions: \( a' = a/a \), \( a \cdot b = a'/b' \), and \( a + b = (a/b)' \).

III. There exists in \( K \) a unique element \( 0 \), such that, if \( f(x) \) is any function definable in terms of and elements of \( K \), we have, for any \( x \) in \( K \),

   \[ f(x) = f(0')x + f(0)x'. \]

**Theorem 1.** \( 0'' = 0 \).

**Proof:** From III, and the preceding definitions, we have

(1) \[ x = 0'x + 0x' = [(0'x)/(0x')]'; \]

in particular

* This is the smallest set of postulates for a Boolean Algebra yet given.
0 = [(0'0)/(00')]'.

Thus there exists in $K$ an element 1, such that

\[ 0 = 1'. \]

From (1) above

\[ 1 = 0'1 + 01. \]

and

\[ 1 = 1''1 + 1'1'. \]

Letting $f(x) = x''x + x'x'$ in III, we have

\[ x''x + x'x' = (0''0' + 0'0'')x + (0'0 + 0'0')x'; \]

in particular

\[ 1 = 1''1 + 1'1' = (0''0' + 0'0'')1 + (0'0 + 0'0')1'. \]

This becomes

\[ 1 = 0''1 + 0'1', \]

since letting $f(x)$ equal $x''$ and $x'$ respectively in III yields

\[ x'' = 0''x + 0''x', \text{ and } x' = 0''x + 0'x'; \]

in particular

\[ 0'' = 0''0' + 0'0'', \text{ and } 0' = 0'0 + 0'0'. \]

Letting $f(x) = 0'''x + x''x'$ in III, we have

\[ 0'''x + x''x' = (0'''0' + 0'''0'')x + (0'0 + 0'0')x'; \]

in particular

\[ 1 = 0'''1 + 1''1' = (0'''0' + 0'''0'')1 + (0'0 + 0'0')1'. \]

This becomes

\[ 1 = 0'''1 + 0''1', \]

since

\[ 0''' = 0'''0' + 0'''0'', \text{ and } 0'' = 0''0 + 0'0'. \]

From III, $x''' = 0'''x + 0''x'$; thus $1''' = 0'''1 + 0''1'$, and $1 = 1''$. Hence $1' = 1'''$ and $0 = 0''$.

**Theorem 2.** $x''' = x$. 

\[ P \text{roof: } x'' = x''' + 0'''x' = 0'x + 0x' = x. \]

From Theorem 1 and the definition of 1, we have \(0' = 1\), and thus
\[ 0/0 = 0' = 1, \quad \text{and} \quad 1/1 = 1' = 0. \]

**Theorem 3.** \(1/0 = 0/1 = 1\).

**Proof:** From III and the definitions, we have
\[ 1 = 0'1 + 01' = 0/0 + 1/1 = 1 + 0 = (1/0)'; \]
\[ 0 = 1/0. \]
\[ 0' = 0''0 + 0'0' = 1/1 + 0/0 = 0 + 1 = (0/1)'; \]
\[ 0 = 0/1. \]

**Theorem 4.** \(0 + 0 = 0, \quad 1 + 0 = 1, \quad 0 + 1 = 1, \quad 1 + 1 = 1, \quad 00 = 0, \quad 10 = 0, \quad 01 = 0, \quad \text{and} \quad 11 = 1. \)

**Proof:** These equations follow immediately upon using the results of the preceding theorems in the definitions of + and .

In the following theorems the equations are obtained by letting \(f(x)\) equal the left-hand side.

**Theorem 5.** \(1x = x = x 1 = 0 + x = x + 0.\)

**Proof:**
\[ 1x = (1 1)x + (1 0)x' = 1x + 0x', \]
\[ x = 1x + 0x', \]
\[ x1 = (1 1)x + (0 1)x' = 1x + 0x', \]
\[ 0 + x = (0 + 1)x + (0 + 0)x' = 1x + 0x', \]
\[ x + 0 = (1 + 0)x + (0 + 0)x' = 1x + 0x'; \]

since all five are equal to \(1x + 0x'\), the theorem follows.

**Theorem 6.** \(0x = x0 = 0 = xx'.\)

**Proof:**
\[ 0x = (0 1)x + (0 0)x' = 0x + 0x', \]
\[ x0 = (1 0)x + (0 0)x' = 0x + 0x', \]
\[ 0 = 0x + 0x', \]
\[ xx' = (1 1')x + (0 0')x' = 0x + 0x'; \]

since all four are equal to \(0x + 0x'\), the theorem follows.

**Theorem 7.** \(1 + x = x + 1 = 1 = x + 1x'.\)

**Proof:**
\[ 1 + x = (1 + 1)x + (1 + 0)x' = 1x + 1x', \]
\[ x + 1 = (1 + 1)x + (0 + 1)x' = 1x + 1x', \]
\[ 1 = 1x + 1x', \]
\[ x + x' = (1 + 1')x + (0 + 0')x' = 1x + 1x'; \]
since all four are equal to $1x + 1x'$, the theorem follows.

**Theorem 8.** $ax = xa$.

*Proof:* $ax = (a + 1)x + (a + 0)x' = (1 + a)x + (0 + a)x' = xa$.

**Theorem 9.** $a + x = x + a$.

*Proof:* $a + x = (a + 1)x + (a + 0)x' = (1 + a)x + (0 + a)x' = x + a$.

**Theorem 10.** $x + bc = (x + b)(x + c)$.

*Proof:* $(x + b)(x + c) = (1 + b)(1 + c)x + (0 + b)(0 + c)x' = (1 + b)c x + (0 + b)c x' = x + bc$.

**Theorem 11.** $xb + xc = x(b + c)$.

*Proof:* $xb + xc = (1b + 1c)x + (0b + 0c)x' = (b + c)x + 0x' = x(b + c)$.

The postulates we have given are known to be true in a Boolean Algebra, therefore they are necessary. We shall show that they are sufficient by showing that Huntington's postulates are derivable from them.

2. **Huntington's Postulates and their Derivation.** The following is Huntington's set of postulates; to each is appended a brief indication of its derivation from those of our set.

1. (a) *If a and b are elements of K, then a + b is an element of K.*

   *By definition, $a + b = (a/b)' = (a/b)/(a/b);$ by Postulate I, if $a$ and $b$ are elements of $K$, $a/b$ is an element, and $a + b$ is an element.*

   (b) *If a and b are elements of K, then a · b is an element of K.*

   *By definition, $a · b = a'/b' = (a/a)/(b/b)$ is an element of $K$ as above.*

2. (a) *There exists an element 0 in K such that $a + 0 = a$.*

   (b) *There exists an element 1 in K such that $a · 1 = a$.*

Theorem 5.

3. (a) *If a, b, a + b, b + a belong to K, then $a + b = b + a$.*

Theorem 9.

(b) *If a, b, ab, ba belong to K, then $ab = ba$.*

Theorem 8.

4. (a) *If a, b, c, bc, a + bc, a + b, a + c, and $(a + b)(a + c)$ belong to K, then $a + bc = (a + b)(a + c)$.*

Theorem 10.
(b) If \(a, b, c, b+c, a(b+c), ab, ac,\) and \(ab+ac\) belong to \(K,\) then \(ab+ac=a(b+c).\)

Theorem 11.

5. If 0 and 1 exist and are unique, then for every element \(a\) belonging to \(K\) there exists an element \(a'\) in \(K\) such that \(a+a'=1\) and \(aa'=0.\)

Theorems 1, 6, and 7.

6. There are at least two distinct elements in \(K.\)

Postulate I.

3. A Two Element Boolean Algebra. A set of postulates for a two element Boolean Algebra can be obtained by changing Postulate I to: "\(K\) contains two and only two elements."

4. Independence Examples:

1. \(K = \{\alpha\}, \quad \alpha/\alpha = \alpha.\)

2. \(K = \{0, \beta\}, \quad \begin{array}{c|cccc} \wedge' & 0 & \beta & \gamma & \delta \\
0 & \delta & \gamma & \beta & 0 \\
\beta & \gamma & \gamma & 0 & 0 \\
\gamma & \beta & 0 & \beta & 0 \\
\delta & 0 & 0 & 0 & 0 \end{array}.\)

3. \(K = \{\alpha, \beta\}, \quad \begin{array}{c|cc} \wedge' & \alpha & \beta \\
\alpha & \alpha & \alpha \\
\beta & \alpha & \alpha \end{array}.\)