AN INTEGRAL EQUATION

Let $D_{H_1}$ denote a domain containing $H_2$ such that $\overline{D_{H_1}} \cdot (K + D_{K_1}) = 0$. Let $D_{K_2}$ denote a domain containing $K_2$ and such that $\overline{D_{K_2}} \cdot (H + D_{H_1} + D_{H_2}) = 0$. This process may be continued and $D_H = \sum D_{H_n}$ and $D_K = \sum D_{K_n}$ are two mutually exclusive domains covering $H$ and $K$ respectively.

THE UNIVERSITY OF TEXAS

ON AN INTEGRAL EQUATION WITH AN ALMOST PERIODIC SOLUTION

BY B. LEWITAN

We assume that the function $f(x)$ is almost periodic in the sense of H. Bohr and that the functions $E(\alpha)$, $\alpha E(\alpha)$ are absolutely integrable in $[-\infty, \infty]$

THEOREM. If all real zeros of the function

$$\gamma(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(u)e^{-i\alpha u} du$$

have integer multiplicities and only two limit points $\infty$, $\alpha^*$, then every solution $\phi(x)$ of the equation

$$\int_{-\infty}^{\infty} E(\xi - x) \cdot \phi(\xi) d\xi = f(x)$$

which is uniformly continuous and bounded in $[-\infty, \infty]$ is almost periodic.

PROOF. Without loss of generality we may assume that the finite limit point $\alpha^*$ has the value $0$; otherwise we multiply equation (1) by $e^{-i\alpha^*x}$.

Putting

$$f_n(x) = \frac{3}{2\pi} \int_{-\infty}^{\infty} f \left( x + \frac{2u}{n} \right) \frac{\sin^4 u}{u^4} du,$$

we obtain

$$\int_{-\infty}^{\infty} E(\xi) \phi_n(\xi + x) d\xi = f_n(x).$$
where
\[
\phi_n(t) = \frac{3}{2\pi} \int_{-\infty}^{\infty} \phi(t + \frac{2u}{n}) \frac{\sin^4 u}{u^4} \, du.
\]

If \(v_n(\alpha)\) denotes the generalized Fourier transform of \(\phi_n(t)\), then, in our case, \(v_n(\alpha)\) is a linear function for \(\alpha > 2n\) and \(\alpha < -2n\).

The functions \(f_n(x)\) and \(\phi_n(x)\) are differentiable and the derivative of \(\phi_n(x)\) is bounded. The function \(E(\xi)\) being absolutely integrable, we therefore obtain
\[
\int_{-\infty}^{\infty} E(\xi)\phi_n'(\xi + x) \, d\xi = f_n'(x).
\]

Putting
\[
\lambda_\epsilon(\alpha) = \begin{cases} 
1 & \text{for } |\alpha| \leq \epsilon, \\
(2 - |\alpha|/\epsilon)^2 (2 |\alpha|/\epsilon - 1) & \text{for } \epsilon \leq |\alpha| \leq 2\epsilon, \\
0 & \text{for } |\alpha| \geq 2\epsilon,
\end{cases}
\]

and
\[
\tau_\epsilon(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda_\epsilon(\alpha) e^{-iu\alpha} \, d\alpha,
\]

\[
f_{n,\epsilon}(x) = f_n'(x) - \int_{-\infty}^{\infty} f_n'(x + u) \tau_\epsilon(u) \, du,
\]

\[
\phi_{n,\epsilon}(x) = \phi_n'(x) - \int_{-\infty}^{\infty} \phi_n'(x + u) \tau_\epsilon(u) \, du,
\]

we obviously have
\[
\int_{-\infty}^{\infty} E(\xi) \cdot \phi_{n,\epsilon}(\xi + x) \, d\xi = f_{n,\epsilon}(x).
\]

If \(v_{n,\epsilon}(\alpha)\) and \(u_{n,\epsilon}(\alpha)\) are generalized Fourier transforms of \(\phi_{n,\epsilon}(x)\) and \(f_{n,\epsilon}(x)\), then the relation \(\dagger\)


\(\dagger\) S. Bochner, loc. cit.
\[ \gamma(\alpha) \nu_{n,\epsilon}(\alpha) = d u_{n,\epsilon}(\alpha) \]

holds.

It follows from the construction of the function \( \lambda_n(\alpha) \) that the function \( \gamma(\alpha) \) has a finite number of zeros in those intervals where \( \nu_{n,\epsilon}(\alpha) \) is not linear. Consequently, by a result of S. Bochner, the function \( \phi_{n,\epsilon}(x) \) is almost periodic in the sense of H. Bohr.

When \( \epsilon \to 0 \), \( \phi_{n,\epsilon}(x) \) converges to \( \phi_n'(x) \) uniformly in \([-\infty, \infty]\). This follows from

\[ \int_{-\infty}^{\infty} \phi_n'(x + u) \tau_{\epsilon}(u) du = \epsilon \int_{-\infty}^{\infty} \phi_n(x + u) \frac{\tau_{\epsilon}'(u)}{\epsilon} du \leq \epsilon M \to 0, \]

where \( M \) is a constant. Hence, \( \phi_n'(x) \) is almost periodic in the sense of H. Bohr. But \( \phi_n(x) \) is bounded. Therefore, by the theorem of Bohr, \( \phi_n(x) \) is also almost periodic. Finally, \( \phi(x) \) being uniformly continuous, the sequence \( \phi_n(x) \) converges to \( \phi(x) \) uniformly in \([-\infty, \infty]\) as \( n \to \infty \), and \( \phi(x) \) is almost periodic itself.

We note that the assertion of the theorem remains valid if, more generally, the limit points of the zeros of \( \gamma(\alpha) \) are isolated; it is also possible to drop the assumption that the zeros have integer multiplicities.

Mathematical Institute,
University A. M. Gorki,
Charkow, U.S.S.R.

† S. Bochner, loc. cit.