PROOF OF THE NON-ISOMORPHISM OF TWO COLLINEATION GROUPS OF ORDER 5184*

BY F. A. LEWIS

Introduction. Let $S$ denote the collineation

$$\rho x_r = e^{r^{-1}}x'_r, \quad (r = 1, \ldots , n), \quad \epsilon = \cos \left(\frac{2\pi}{n}\right) + i \sin \left(\frac{2\pi}{n}\right),$$

and $T$ the collineation

$$\rho x_r = x'_{r+1}, \quad (r = 1, \ldots , n), \quad x'_{n+1} \equiv x'_1.$$

The abelian group $\{S, T\}$ of order $n^r$ is invariant under a group† $C_n$ of order

$$n^r \left(1 - \frac{1}{p_1^2}\right) \left(1 - \frac{1}{p_2^2}\right) \cdots \left(1 - \frac{1}{p_m^2}\right),$$

where $p_1, p_2, \ldots , p_m$ are the distinct prime factors of $n$. The order of $C_6$ is 5184.

Winger‡ has discussed briefly the monomial group of order $(r+1)!n^r$ that leaves invariant the variety

$$x_0^n + x_1^n + x_2^n + \cdots + x_r^n = 0.$$

This group is generated by the symmetric group of degree $r+1$ and an abelian group of order $n^r$ in canonical form. For $r = 3$ and $n = 6$ there results a group $G$ of order 5184 which has been treated by Musselman.§ The purpose of this note is to prove that $G$ and $C_6$ are not simply isomorphic. The proof consists in showing that the number of collineations of period 2 in $G$ exceeds the number of collineations of period 2 in $C_6$.

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† In fact, $C_n$ is the largest collineation group in $n$ variables containing $\{S, T\}$ invariantly, the coefficients and variables being in the field of complex numbers. (Author's dissertation, Ohio State University, 1934.)
Proof of the Non-Isomorphism of \( G \) and \( C_6 \). The group \( C_6 \) is generated by \( \{S, T\} \) and the two collineations

\[
V: \quad \rho x_r = \sum_{c=1}^{6} \epsilon^{(r-1)(c-1)} x'_c, \quad (r = 1, \ldots, 6),
\]

\[
W: \quad \rho x_r = \epsilon^{-(r-1)/2} x'_r, \quad (r = 1, \ldots, 6),
\]

satisfying the following relations:

\[
V^4 = W^{12} = 1, \quad V^2W = WV^2, \quad V^{-1}SV = T^{-1}, \quad W^{-1}SW = S, \quad (VW)^3 = V^2 = (WV)^3, \quad W^6 = S^3, \quad V^{-1}TV = S, \quad W^{-1}TW = S^{-1}T.
\]

The order of \( H = \{V, W\} \) is 576. This group may be constructed by the following chain of invariant subgroups and an independent proof that the order of \( C_6 \) is 5184 follows readily.

\[
H = \{V, G_{288}\}, \quad G_{288} = \{W^4VW^3V^3, G_{96}\}, \quad G_{96} = \{W^2, G_{32}\}, \quad G_{32} = \{W^2(W^2V)^3, G_{16}\}, \quad G_{16} = \{(W^2V)^3V, G_4\}, \quad G_4 = \{S^3, T^3\}.
\]

Since \( G_4 \) is contained in \( \{S, T\} \) which is invariant under \( H \), the order of \( C_6 \) is \( 576 \cdot 36/4 = 5184 \).

If \( Q \), of order 144, represents the quotient group of \( C_6 \) with respect to \( \{S, T\} \), each element of \( Q \), being a co-set of \( C_6 \), represents 36 collineations of \( C_6 \) that transform \( \{S, T\} \) into itself according to the same isomorphism of \( \{S, T\} \) with itself.* There are 24 collineations \( S'T^a \) of period 6 in \( \{S, T\} \); if \( S \) is transformed into a particular \( S'T^a \), the collineation \( S'T^m \) into which \( T \) is to be transformed may be selected in six ways. Let \( K \) represent a class of 144 collineations of \( C_6 \) corresponding to the 144 distinct possible sets \( (j, k, l, m) \). That is, \( K \) contains one and only one collineation from each of the 144 augmented co-sets of \( C_6 \). The square of \( A \cdot S'T^a \), an arbitrary collineation of the class \( K \) from the co-set to which \( A \) belongs, may be expressed in the form \( A^2S^mT^n \) and hence is of period 2 only if \( A^2 \) is in \( \{S, T\} \). That is, a necessary condition that \( A \cdot S'T^a \) be of period 2 is that \( A^2 \) be commutative with both \( S \) and \( T \). Among any class \( K \) there are only 8 collineations \( B \) such that the corresponding sets of values \( (j, k, l, m) \) satisfy the congruences arising from the conditions that \( B^2 \) transform \( S \) into \( S \) and \( T \) into \( T \).

* It may easily be proved that the 36 collineations of \( \{S, T\} \) are the only collineations in six variables commutative with both \( S \) and \( T \).
The following table shows 8 such collineations, their squares, and the collineations of \( \{S, T\} \) which multiply these 8 collineations on the right to form collineations of \( C_6 \) of period 2. The numbers in the last column show the total number of collineations of \( C_6 \) of period 2 corresponding to each \( B \) of \( K \). Thus it is seen that \( C_6 \) contains just 99 collineations of period 2.

It is easily shown that \( G \) contains more than 99 collineations of period 2 and hence \( G \) and \( C_6 \) are not simply isomorphic.

<table>
<thead>
<tr>
<th>( W^2 )</th>
<th>( W^2 = S^2 )</th>
<th>( T^3, S^3 T^3 )</th>
<th>( X^2 = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S^2 )</td>
<td>( S^2 = 1 )</td>
<td>( 1, T^3, S^3 T^3 )</td>
<td>( 3 )</td>
</tr>
<tr>
<td>( U^2 = V^{-1} W^2 V )</td>
<td>( U^2 = T^3 )</td>
<td>( S^3, S^3 T^3 )</td>
<td>( 2 )</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
W^2 &\equiv -1 & S^2 &\equiv 0 & T^3, S^3 T^3 &\equiv 2 \\
S^2 &\equiv 0 & S^2 &\equiv 0 & 1, T^3, S^3 T^3 &\equiv 3 \\
U^2 &\equiv 0 & U^2 &\equiv 0 & S^3, S^3 T^3 &\equiv 2 \\
V^2 &\equiv 0 & X^2 &\equiv 1 & S^3 T^3 \text{ where } (j, k) \text{ satisfies the congruence } 3(j+k) \equiv 0 \pmod{6} &\equiv 18 \\
R X &\equiv 0 & (RX)^2 &\equiv 1 & S^3 T^3 \text{ where } (j, k) \text{ satisfies the congruence } 3(j+k) \equiv 0 \pmod{6} &\equiv 36 \\
R W^3 &\equiv W^3 & (RW^3)^2 &\equiv W^3 & S^3 T^3 \text{ where } (j, k) \text{ satisfies the congruence } 3(j+k) \equiv 0 \pmod{6} &\equiv 2 \\
R U^3 &\equiv U^3 & (RU^3)^2 &\equiv T^3 & S^3 T^3 \text{ where } (j, k) \text{ satisfies the congruence } 3(j+k) \equiv 0 \pmod{6} &\equiv 18 \\
\end{align*}
\]

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