we have

\[ \lim_{k \to \infty} \delta_i^{(k)} = \prod_{j} d_{i,j} \prod_{j} D_{i,j}. \]

Now

\[ |\delta_i^{(k)} - \delta| = \delta \left| 1 - \frac{\prod_j D_{i,j}}{\prod_j D_{i,j}^{(k-1)}} \right|. \]

But the quantity on the right approaches zero, so that \( \delta_i^{(k)} \to \delta \)
as \( k \to \infty \). We thus have (1), and the theorem is proved.

1. Introduction. It is a well known fact that all planes which meet four general lines of \( S_4^* \) are met by a fifth line. The remarkable configuration determined by five such "associated lines" is discussed in a number of places in the literature.† In the present paper an involutorial line transformation suggested by the figure of five associated lines is discussed, both as a line involution in \( S_4 \), and as a point involution on a certain \( V_6 \) in \( S_9 \). In §§2–6 the involution is treated at some length by purely synthetic methods. The final section (§7) contains a brief analytic treatment, including the equations of the involution, and the equations of the invariant and singular elements. The involu-

* We shall use the conventional symbol \( S_m \) to indicate a linear space of dimension \( m \). A variety of order \( r \) and of dimension \( m \) we shall designate by the symbol \( V_{mr} \).


tion appears to be an important one in that most of the loci connected with it are of considerable interest in the geometry of hyperspace.

2. Representation of Five Associated Lines in $S_9$. Stephanos has shown* that five associated lines have their ten Grassmann coordinates linearly related. If we regard the ten Grassmann coordinates of the lines of $S_4$ as point coordinates in $S_9$, † and recall that the five quadratic identities existing among these coordinates define a $V_8^6$ in $S_9$ ‡ whose points are in 1:1 correspondence with the lines of $S_4$, this result may be interpreted as follows: The images in $S_9$ of five associated lines of $S_4$ are the five points in which a general $S_3$ meets $V_8^6$. In fact, from this as a definition of five associated lines the incidence property mentioned above follows at once. For consider four general lines, $L_1, L_2, L_3, L_4$, of $S_4$, and any plane, $P$, meeting each of these lines. On $V_8^6$ we have the four independent points $l_1, l_2, l_3, l_4$, which are the images of the four lines $L_1, L_2, L_3, L_4$. § Now the $S_8$ which intersects $V_8^6$ in the $V_8^6$ representing the special complex of lines which meet the plane $P$ contains $l_1, l_2, l_3, l_4$. It therefore contains $l_5$, the fifth point in which the $S_3$ determined by $l_1, l_2, l_3, l_4$ intersects $V_8^6$. Hence $l_5$ is also the image of a line $L_5$, which meets the plane $P$.

3. Definition of the Involution. Let there be given three general lines, $L_1, L_2, L_3$, of $S_4$ and let the transform of any line $L$ be the line $L'$ which forms with $L_1, L_2, L_3$, and $L$ a set of five associated lines. On $V_8^6$ this becomes a point involution defined as follows: Let there be given three points, $l_1, l_2, l_3$, on $V_8^6$ such that the plane, $\pi$, which they determine meets $V_8^6$ only in these three points. Then the transform of any point, $l$, is the fifth point, $l'$, in which the $S_3$ determined by $l_1, l_2, l_3$, and $l$ meets $V_8^6$.

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‡ See §7 for the analytic definition of the Grassmann coordinates, and of the five quadratic identities.
§ Throughout this paper we shall represent configurations in $S_4$ by capital letters, and their images in $S_9$ by small letters.
To determine the order of the involution consider a general pencil of lines, $R$, of $S_4$. The image of this pencil on $V_6^\delta$ is a line, $r$. The $\infty^1$ spaces $S_5$ determined by $\pi$ and the respective points of $r$ all lie in the $S_4$ determined by $\pi$ and $r$. This $S_4$ meets $V_6^\delta$ in a curve of order five. The line $r$ is a point of this curve of intersection. The other component is a rational normal quartic curve, passing through the three points $l_1, l_2, l_3$ in $\pi$. This quartic curve is the transform of $r$ in the involution on $V_6^\delta$, hence the involution in $S_4$ is of order four, transforming a pencil of lines into a ruled quartic surface, three of whose generators are the fundamental lines $L_1, L_2, L_3$.

4. Invariant Elements. There are $\infty^4$ varieties $V_4^\delta$ on $V_6^\delta$, these being the representations in $S_9$ of the lines of the $\infty^4$ spaces $S_5$ in $S_4$.* Each of these $V_4^\delta$'s lies in an $S_5$. One and only one of these $S_5$'s passes through a general point of $S_9$, while $\infty^2$ of them pass through each point of $V_6^\delta$. In particular the $\infty^2$ spaces $S_5$ corresponding to the points of $\pi$ cut $V_6^\delta$ in $V_4^\delta$'s each of which contains the point, $t$, which is the representation in $S_9$ of the common transversal, $T$, of the lines $L_1, L_2, L_3$, in $S_4$. These $V_4^\delta$'s are thus representations of the lines of the $\infty^2$ spaces $S_5$ through $T$ in $S_4$.

Consider now a point, $l$, and its transform, $l'$, in the involution on $V_6^\delta$, and consider further the $S_5$ corresponding to the point, $q$, in which the line joining $l$ and $l'$ intersects $\pi$. Any other point, $m$, on the $V_4^\delta$ cut from $V_6^\delta$ by this $S_5$ must have for its transform, $m'$, another point of the same $V_4^\delta$, namely the second point in which the line joining $q$ and $m$ intersects the $V_4^\delta$. Thus the involution on $V_6^\delta$ leaves invariant as a whole, though not point by point, the $\infty^2$ varieties $V_4^\delta$ corresponding to the points of $\pi$. This implies that in $S_4$ the transform of any line $L$ lies in the $S_5$ determined by $L$ and the transversal, $T$, of the three lines $L_1$, $L_2$, $L_3$.

The invariant points in the involution on $V_6^\delta$ are the points of contact of the tangents to $V_6^\delta$ which meet $\pi$. A triple infinity of these tangents pass through each point, $q$, of $\pi$. They are in fact the $\infty^3$ tangents to the $V_4^\delta$ corresponding to the point $q$; and the locus of their points of contact is the $V_4^\delta$ cut from the $V_4^\delta$ by the polar $S_4$ of $q$ as to the $V_4^\delta$ (the polar $S_4$ being deter-

* Todd, loc. cit.
mined in the $S_8$ containing $q$ and the corresponding $V^2_8$). Thus in $S_4$ the invariant lines of the involution form a linear complex in each $S_8$ through $T$.

On $V^8_6$ there are $\infty^5$ planes with the property that they contain three non-concurrent lines which determine with the respective points $l_1, l_2, l_3$ planes lying entirely on $V^6_8$. (These planes are the representations in $S_8$ of the fields of lines lying in planes which meet $L_1, L_2, L_3$.) Each of the three lines in such a plane, $\sigma$, is singular in the involution, its image being the plane which joins it to the corresponding point, $l_4$, plus two lines joining $l_i$ to the other two points, $l_j$ and $l_k$. The image of $\sigma$ itself is the residual $V^4_2$ cut from $V^8_6$ by the $S_8$ determined by $\pi$ and $\sigma$. This $V^4_2$ is composite, consisting of the three planes which are the images of the three singular lines in $\sigma$, and a fourth plane, $\sigma'$. Now $\sigma'$ must meet in a line each of the other planes composing $V^8_6$, since it lies on $V^6_8$ and meets in a line the $S_8$ determined by $\pi$ and each of these planes. But this requires either that $\sigma'$ coincide with $\sigma$, or that $\sigma$ and the planes composing $V^8_6$ lie in an $S_4$ instead of in an $S_8$. The latter is impossible, hence $\sigma$ and $\sigma'$ coincide, and the non-singular points of $\sigma$ are invariant. Conversely, every invariant point on $V^8_6$ lies in a plane $\sigma$, namely: the plane common to the three $V^4_2$'s determined by the invariant point and $l_1, l_2, l_3$, respectively.

Thus in $S_4$ the invariant lines are the $\infty^5$ lines which lie in planes meeting $L_1, L_2, L_3$. Hence in a general $S_8$ of $S_4$ the invariant lines form a tetrahedral complex, whose fundamental tetrahedron is determined by the four points in which the $S_8$ intersects $L_1, L_2, L_3$, and $T$. In an $S_8$ through $T$ this quadratic complex breaks up into a general linear complex, as we have already noted, and the special linear complex of lines meeting $T$, the latter complex being singular and not properly invariant.

5. **Singular Elements.** The singular elements in the involution on $V^8_6$, aside from the three fundamental points $l_1, l_2, l_3$, are those points which determine with $\pi$ an $S_8$ meeting $V^8_6$ in a curve instead of in five points. They are of three general types: Those points which lie on lines of $V^6_8$ passing through $l_4$, those

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points which lie on conics of $V_6^t$ passing through $l_i$ and $l_j$, and those points which lie on cubics of $V_6^t$ passing through $l_i, l_j, l_k$.

The last class of points is also the class of points lying on lines of $V_6^t$ which pass through $t$. This is evident when we consider the $S_4$ determined by $t$ and an $S_3$ containing a cubic curve through $l_1, l_2, l_3$. Such an $S_4$ contains not only the cubic curve but also the lines joining $t$ to $l_1, l_2, l_3$. Since these lines lie on $V_6^t$, the $S_4$ thus meets $V_6^t$ in a curve of order greater than five, and hence has a surface in common with $V_6^t$. This surface must be a cubic cone with vertex at $t$, and the given cubic as directrix curve. Hence all points on any cubic through $l_1, l_2, l_3$ lie on lines through $t$.

The three classes of singular points correspond respectively to the following classes of singular lines in $S_4$: Lines meeting $L_i$, lines lying in the $S_3$ determined by $L_i$ and $L_j$, and lines meeting $T$. The images of lines belonging to one or more of these classes may be described as follows: (The symbol $\sim$ is used to mean "is transformed into.")

1. A general line, $L$, meeting $T \sim$ the cubic regulus containing $L_i, L_2, L_3$, and $L$ as generators, and having $T$ as directrix.
2. A general line, $L$, in the $S_3$ determined by $L_i$ and $L_j$, the quadratic regulus determined by $L_i, L_j$, and $L$.
3. A general line, $L$, meeting $L_i \sim$ the pencil determined by $L_i$ and $L$.
4. A general line, $L$, through the intersection of $T$ and $L_i \sim$ the pencil determined by $T$ and $L_i$, together with the quadratic regulus determined by $L_i, L_j$, and the line of the pencil which lies in the $S_3$ determined by $L_j$ and $L_k$.
4.1. A general line, $L$, meeting $T$ and lying in the $S_3$ determined by $L_i$ and $L_j$, the quadratic regulus determined by $L_i, L_j$, and $L$, together with the pencil determined by $L_k$ and the generator of the regulus which passes through the intersection of $T$ and $L_k$.
5. A line, $L$, in the $S_3$ determined by $L_i$ and $L_j$ passing through the intersection of $T$ and $L_i \sim$ the pencil determined by $L_i$ and $L$, the pencil determined by $L_i$ and the line of the first pencil which meets $L_j$, and the pencil determined by $L_k$ and the line of the second pencil which meets $L_k$.
5.1. A general line $L$, in the plane of $T$ and $L_i \sim$ the pencil determined by $L_i$ and $L$, the pencil determined by $L_j$ and the
line of the first pencil which meets \( L_i \), and the pencil determined by \( L_k \) and the line of the first pencil which meets \( L_k \).

6. A general line, \( L \), meeting \( L_i \) and \( L_j \) ~ the pencil determined by \( L_i \) and \( L \), together with the pencil determined by \( L_j \) and \( L \).

6.1. A general line, \( L \), meeting \( L_i \) and lying in the \( S_\delta \) determined by \( L_i \) and \( L_j \) ~ the pencil determined by \( L_i \) and \( L \), together with the pencil determined by \( L_j \) and the line of the first pencil which meets \( L_j \).

7. The transversal, \( T \), ~ the three pencils determined respectively by \( T \) and \( L_i \), \( T \) and \( L_j \), and \( T \) and \( L_k \).

8. The fundamental line, \( L_i \), ~ the special linear complex consisting of all lines which meet the plane determined by \( T \) and \( L_i \).

None of these but the last needs special comment; we verify it in this fashion: Any point which is an image of \( l_i \) in the involution on \( V_\delta^3 \) must be such a point that the \( S_\delta \) which it determines with \( \pi \) contains one of the tangents to \( V_\delta^5 \) at \( l_i \). There are \( \infty^5 \) tangents which can be drawn to \( V_\delta^5 \) at the point \( l_i \). All these points lie in the \( S_\delta \) determined by \( \pi \) and the tangent \( S_\delta \) to \( V_\delta^5 \) at \( l_i \), and hence correspond to lines of a special linear complex in \( S_\delta \). The singular plane of this complex must contain \( L_i \), and must be met by \( L_j \) and \( L_k \). It must therefore be the plane determined by \( T \) and \( L_i \).

6. Images of Linear Systems of Lines. As we have already noted, every line, \( L \), which meets either \( L_1 \), \( L_2 \), or \( L_3 \), is transformed into a pencil containing \( L \). Although such lines are singular they may thus be regarded as invariant also. Hence \( L_1 \), \( L_2 \), and \( L_3 \) must lie upon all the quadratic hypercones formed by the invariant lines which pass through the respective points of \( S_\delta \). Moreover \( L_1 \), \( L_2 \), and \( L_3 \) lie in planes of the same family on each of these hypercones.

Consider now the \( \infty^3 \) lines through a general point of \( S_\delta \). On \( V_\delta^3 \) these are represented by the points of an \( S_\delta \). The transform of this \( S_\delta \) is the residual \( V_\delta^4 \) in which the \( S_\delta \) determined by \( \pi \) and the \( S_\delta \) intersects \( V_\delta^5 \). Now the \( S_\delta \) whose intersection with \( V_\delta^5 \) represents the special complex of lines meeting any plane of the second family (the family not containing \( L_1 \), \( L_2 \), and \( L_3 \)) on the hypercone of invariant lines through the given point in \( S_\delta \), con-
tains \( l_1, l_2, l_3 \) and the \( S_8 \) which represents the totality of lines through the given point. The \( S_8 \) therefore contains the entire \( S_6 \) in which the image \( V_3^4 \) lies; hence all points of \( V_3^4 \) represent lines of \( S_4 \) which meet all planes of the second family on the hypercone of invariant lines. Thus \( V_3^4 \) must represent the totality of lines lying in the planes of the first family on the hypercone, and this system is then the transform of the \( \infty^3 \) lines through a general point of \( S_4 \).

Since the double infinity of lines through a general point in a general \( S_8 \) in \( S_4 \) is included in the totality of all lines through the point, the transform of such a system must be \( \infty^2 \) lines lying in planes of the first family on the hypercone of invariant lines through the point. These lines are in fact the lines which lie in planes of the first family and meet the plane determined by the points where the \( S_8 \) intersects \( L_1, L_2, \) and \( L_3 \). For on \( V_6^5 \) the image of the given system of lines is a plane, and its transform is the residual \( V_6^5 \) in which the \( S_8 \) determined by \( \pi \) and this plane meets \( V_6^5 \). Now the \( S_8 \) which meets \( V_6^5 \) in the \( V_5^8 \) representing the special complex of lines meeting the plane, \( P \), determined in \( S_4 \) by the points where the given \( S_5 \) intersects \( L_1, L_2, \) and \( L_3 \), contains \( l_1, l_2, l_3 \), and the plane which represents the given system of lines. It therefore contains the \( S_8 \) in which the image \( V_2^4 \) lies. Hence all points of this \( V_2^4 \) represent lines meeting the plane \( P \). Since the family of lines represented by this \( V_2^4 \) contains \( \infty^1 \) pencils of lines (one in each plane of the first family on the hypercone), the \( V_2^4 \) must be the rational normal ruled \( V_2^4 \) of \( S_6 \), and not the Veronese surface.

We have already noted that the transform of a general pencil of lines of \( S_4 \) is a rational ruled quartic surface containing \( L_1, L_2, \) \( L_3 \), and two lines of the given pencil, namely the two invariant lines. These two lines are the only generators of the surface which can intersect. This surface, the projection of the rational normal ruled quartic surface of \( S_6 \), must also lie on the hypercone of invariant lines through the vertex of the pencil. In fact this hypercone is the only quadratic primal on which the surface can lie.*

To determine the image of a plane field of lines in \( S_4 \) consider the \( S_8 \) which is determined in \( S_5 \) by \( \pi \) and the plane, \( p \), which

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represents the lines of the given plane field. This $S_5$ intersects $V_5^a$ in the plane $p$, and in a $V_3^b$, the Veronese surface, which is the transform of $p$. Now consider the $S_5$ whose intersection with $V_5^b$ represents the special linear complex of lines meeting any one of the planes determined in $S_4$ by the points where an $S_3$ through the plane field intersects $L_1$, $L_2$, and $L_3$. This $S_5$ contains $l_1$, $l_2$, $l_3$, and the plane $p$. It therefore contains the $S_5$ determined by $\pi$ and $p$, and hence contains $V_3^a$. All points of $V_3^b$ are thus representations of lines of $S_4$ which meet all the planes determined by the respective triads of points in which the $S_3$'s through the plane field intersect $L_1$, $L_2$, and $L_3$. As a point locus this system of lines is a $V_3^b$ having a plane of double points.\textsuperscript{*}

7. Analytic Procedure. Let the three fundamental lines in $S_4$ be determined by the respective pairs of points: $(10000)(01000)$, $(00100)(00010)$, $(00001)(10100)$. The common transversal of these lines is the line joining $(10000)$ and $(00100)$. The Grassmann coordinates of these lines, as read from the matrices of the points which determine each line, are

\[
\begin{array}{cccccccc}
P_{12} & P_{13} & P_{14} & P_{15} & P_{23} & P_{24} & P_{25} & P_{34} & P_{35} & P_{45} \\
L_1 : l_1 : & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
L_2 : l_2 : & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
L_3 : l_3 : & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
T : t : & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{array}
\]

The quadratic identities which define $V_5^a$ are

\[
\begin{align*}
P_{23}P_{45} - P_{24}P_{35} + P_{25}P_{34} &= 0, \\
P_{13}P_{45} - P_{14}P_{35} + P_{15}P_{34} &= 0, \\
P_{12}P_{45} - P_{14}P_{25} + P_{15}P_{24} &= 0, \\
P_{12}P_{35} - P_{13}P_{25} + P_{15}P_{23} &= 0, \\
P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} &= 0.
\end{align*}
\]

The $S_3$ determined in $S_9$ by $l_1$, $l_2$, $l_3$, and a general point $l = (l_{12} \cdots l_{15})$ of $V_3^b$ can be written parametrically:

\[
y = \lambda(l_1) + \mu(l_2) + \delta(l_3) + \rho(l).
\]

This meets $V_3^b$ in the points $l_1$, $l_2$, $l_3$, $l$, and a fifth point $l'$, given by

\[ \lambda = -l_{24}l_{26}(l_{13}l_{45} - l_{29}l_{34}), \]
\[ \mu = l_{24}l_{46}(l_{13}l_{45} - l_{29}l_{34}), \]
\[ \delta = l_{25}l_{46}(l_{13}l_{45} - l_{29}l_{34}), \]
\[ \rho = l_{24}l_{26}l_{45}. \]

(A)

Thus the equations of the transformation are:

\[ kl_{14} = l_{25}l_{26}l_{34}, \quad kl_{34} = l_{24}l_{26}l_{46}, \]
\[ kl_{13} = l_{13}l_{22}l_{45}, \quad kl_{35} = l_{24}l_{26}l_{45}, \]
\[ kl_{12} = l_{14}l_{22}l_{45}, \quad kl_{34} = l_{13}l_{26}l_{45}, \]
\[ kl_{25} = l_{25}l_{46}(l_{14} - l_{34}), \quad kl_{35} = l_{13}l_{46}(l_{12} + l_{23}), \]
\[ kl_{23} = l_{25}l_{46}l_{26}l_{45}, \quad kl_{34} = l_{24}l_{26}l_{45}. \]

(B)

From equations (A) it is evident that the complex of invariant lines is given by

\[ l_{13}l_{45} - l_{25}l_{34} = 0. \]

From (A) it also follows that the image of \( L_1 \) is any line of the special complex \( l_{46} = 0 \); the image of \( L_2 \) is any line of the special complex \( l_{25} = 0 \); and the image of \( L_3 \) is any line of the special complex \( l_{24} = 0 \).

The singular lines are lines of the following systems:

1. Lines for which \( l_{34} = l_{35} = l_{45} = 0 \), (lines meeting \( T \)).
2. Lines for which \( l_{34} = l_{35} = l_{45} = 0 \), (lines meeting \( L_1 \)).
2.1. Lines for which \( l_{12} = l_{15} = l_{25} = 0 \), (lines meeting \( L_2 \)).
2.2. Lines for which \( l_{12} + l_{23} = l_{24} = l_{14} - l_{34} = 0 \), (lines meeting \( L_3 \)).
3. Lines for which \( l_{13} = l_{25} = l_{35} = l_{45} = 0 \), (lines lying in the \( S_3 \) determined by \( L_1 \) and \( L_2 \)).
3.1. Lines for which \( l_{14} = l_{24} = l_{34} = l_{45} = 0 \), (lines lying in the \( S_3 \) determined by \( L_1 \) and \( L_3 \)).
3.2. Lines for which \( l_{12} = l_{23} = l_{24} = l_{25} = 0 \), (lines lying in the \( S_3 \) determined by \( L_2 \) and \( L_3 \)).