LINEAR OPERATIONS ON FUNCTIONS OF BOUNDED VARIATION

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The object of this note is to give a form for the most general linear continuous operation on the space of functions of bounded variation on a finite interval, say \(0 \leq x \leq 1\), the norm of the space being the total variation.

This form is obtained by setting up an equivalent space. For this purpose let \(\mathcal{I}\) be the class of elements \(I\) consisting of any finite number of non-overlapping intervals \(i_1, \ldots, i_n\) of the interval \((0, 1)\). If \((x_p, y_p)\) are endpoints of \(i_p\), define the function of interval sets \(\beta(I) = \sum_{p=1}^{n} [\alpha(y_p) - \alpha(x_p)]\) corresponding to the function \(\alpha(x)\) of bounded variation. Then \(\beta(I)\) is a bounded function on \(\mathcal{I}\). Define \(\|\beta\|\) in the usual way as the least upper bound of \(|\beta(I)|\) for \(I\) on \(\mathcal{I}\). Then the space \(\mathcal{B}\) of additive set functions \(\beta\) thus normed is equivalent to the space \(\mathcal{A}\) of functions \(\alpha(x)\) of bounded variation with \(\|\alpha\| = V\alpha = \int_0^1 |d\alpha|\), * for obviously \(\|\beta\| \leq \|\alpha\| \leq 2\|\beta\|\). Further, if \(\alpha_1\) corresponds to \(\beta_1\) and \(\alpha_2\) to \(\beta_2\), then \(\beta_1 + \beta_2\) corresponds to \(\alpha_1 + \alpha_2\) and \(c\beta\) to \(c\alpha\), and conversely.

It is now an easy matter to determine the most general linear functional operation on the space \(\mathcal{B}\). Following the lines of reasoning of my paper On bounded linear functional operations,† one finds that for any linear continuous operation \(L\) on the space \(\mathcal{B}\) there exists an additive function \(\gamma\) of sets \(E\) of elements \(I\), such that \(L(\beta) = \int \gamma d\gamma\), the integral being of the \(L\) or \(S\) type as defined in the paper quoted, and extended over the class of all subsets of elements of \(\mathcal{I}\). Because of the relationship between the functions \(\beta\) and \(\alpha\) this gives the most general linear operation in the space \(\mathcal{A}\).

It might be noted that a similar reasoning applies to the set of interval functions \(\alpha(i)\) where \(\sum_{p=1}^{n} \alpha(i_p) = \beta(I)\) is a bounded function on \(\mathcal{I}\); or, more generally, that a similar result holds in the space of bounded functions on a general range, with norm the least upper bound of the absolute value of the function on the range.

* Note that in the space \(\mathcal{A}\) two functions for which \(V(\alpha_1 - \alpha_2) = \int |d(\alpha_1 - \alpha_2)| = 0\) are regarded as equivalent. To obtain uniqueness, the condition \(\alpha(0) = 0\) can be added. If we wish that \(\|\alpha\| = 0\) imply \(\alpha = 0\) for all \(x\), we may choose \(\|\alpha\| = |\alpha(0)| + V\alpha\). The space \(\mathcal{B}_t\) corresponding is defined by \(\beta_t(I) = \alpha(0) + \sum_{p=1}^{n} [\alpha(y_p) - \alpha(x_p)] = \alpha(0) + \beta(I)\) and \(\|\beta_t(I)\| = \|\alpha(0)\| + \|\beta(I)\|\). Reasoning similar to the above can be carried through in this case also.