NOTE ON DEDUCED PROBABILITY DISTRIBUTIONS

R. VON MISES

In this Bulletin, December, 1936, A. H. Copeland* resumed the study of the problem first suggested by H. Poincaré: How can the fact of uniform probability distribution, which we meet so frequently in different games of chance, be explained? Recently E. Hopf devoted a profound essay† to this question and he has just published a short note‡ dealing with his principal results. I want to contribute a quite simple remark which seems to show how far the results are independent of the particular form of dynamical equations.

We assume that there exists a density function \( f(x) \) for the one-dimensional variable \( x \), such that \( \int_a^b f(x)\,dx \) denotes the probability that the value of \( x \) falls in the interval \( (a, b) \) and \( \int_{-\infty}^{\infty} f(x)\,dx = 1 \). If between \( x \) and \( y \) there is established a one-to-one correspondence

\[
y = y(x), \quad x = x(y),
\]

the given density function \( f(x) \) leads to a new density function \( g(y) \) defined by

\[
g(y) = f(x) \frac{dx}{dy}.
\]

The integral \( \int_a^b g(y)\,dy \) gives, of course, the probability that \( y \) belongs to the interval \( (a, b) \) and \( \int_{-\infty}^{\infty} g(y)\,dy = 1 \).

Now we suppose \( y \) to be an "angular" variable, that is, instead of \( y \) we consider the new variable:

\[
\eta = y - \lfloor y \rfloor, \quad 0 \leq \eta < 1,
\]

and try to determine the probability distribution \( \phi(\eta) \) of \( \eta \). Evidently, if \( \nu \) is a positive or negative integer, the probability density of \( \eta \) is given by

\[
\phi(\eta) = \sum_{\nu} g(\eta + \nu) = \sum_{x} f(x_{\nu}) \left( \frac{dx}{dy} \right)_{x=x_{\nu}}; \quad x_{\nu} = x(\eta + \nu).
\]

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† Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 13 (1934).
We may suppose the transformation ratio $dx/dy > 0$. In this case the consecutive values $\cdots, x(-2), x(-1), x(0), x(1), x(2), \cdots$ define an infinite set of intervals corresponding to intervals of length 1 on the $y$-axis. Let $\eta'$ and $\eta''$ be two values of $\eta$; then the corresponding values $x' = x(\eta' + \nu)$ and $x'' = x(\eta'' + \nu)$ fall in the same interval $(x(\nu), x(\nu + 1))$. Therefore, the difference between the values of the products

$$f(x') \left( \frac{dx}{dy} \right)_{x=x'}, \quad f(x'') \left( \frac{dx}{dy} \right)_{x=x''},$$

is less than or equal to the variation of the product $f \cdot dx/dy$ through the interval $(x(\nu), x(\nu + 1))$, and the difference between the two values of the sum (4) for $\eta'$ and $\eta''$ does not exceed the value of the total variation of the same product. Hence, our theorem follows:

The maximum difference between two values of the deduced probability density $\phi(\eta)$ is less than or equal to the total variation of the product of initial density $f(x)$ by the transformation ratio $dx/dy$.

If we consider an infinite set of similar problems where the initial distribution $f(x)$ remains unchanged and the transformation ratio is multiplied by a parameter $\lambda$, then the deduced distribution $\phi(\eta)$ approaches uniformity as the parameter $\lambda$ approaches 0 and the functions $f(x)$ and $dx/dy$ are of finite variation.

The mechanical example mentioned by Copeland and by Hopf consists in a system rotating about a vertical axis and subjected to friction forces which depend on the instantaneous angular velocity $\omega$. The dynamical equation is given by

$$\frac{d\omega}{dt} = - r(\omega).$$

Let $x$ be the initial value $\omega_0$ of $\omega$. Until the system comes to rest, a point at the distance 1 from the axis will travel a distance which may be designed by $2\pi y$. Then it follows from (5) that

$$2\pi y = \int_0^\infty \omega dt = \int_0^\infty \frac{\omega d\omega}{r(\omega)}, \quad \frac{dx}{dy} = 2\pi \frac{r(x)}{x}.$$ 

Copeland supposes the friction $r(x)$ to be proportional to a parameter $\lambda$, but the distribution $f(x)$ to be independent of $\lambda$. In this case it is clear that, in consequence of (6), $f(x) \cdot dx/dy$ approaches zero with $\lambda \to 0$, and our theorem shows that the asymptotic value of $\phi(\eta)$ is a constant.
On the other hand, Hopf considers the initial distribution to be given in the form \( f(x) = f_1(x - \lambda) \), where \( f_1 \) is a function of one variable and \( \lambda \) a parameter. Moreover, he supposes that

\[
\lim_{x \to \infty} \frac{r(x)}{x} = 0.
\]

We find

\[
f(x) - \frac{dx}{dy} = 2\pi f_1(x - \lambda) \cdot \frac{r(x)}{x} = 2\pi f_1(x) r(\lambda + z)/(\lambda + z),
\]

which approaches zero, according to (7), as \( \lambda \) increases. If the functions \( f_1 \) and \( r/x \) are of finite variation, it follows from our theorem that \( \phi(\eta) \) approaches uniformity.

It is a quite different question to decide whether the foregoing investigation is or is not sufficient to explain the fact of the nearly perfect uniformity of distribution in a particular case of a real game. Let us consider a sort of roulette consisting of a billiard ball which runs in a smooth circular channel subjected to a constant resistance \( r(\omega) = c \); the number of revolutions is found to vary from about 8.1 to 12.1. Our equation (6) gives

\[
2\pi y = \frac{x^2}{2c}, \quad \frac{dx}{dy} = \frac{2\pi c}{x} = \left( \frac{\pi c}{y} \right)^{1/2}, \quad x = 2(\pi cy)^{1/2}.
\]

Therefore \( x \) varies from \( 9(0.4 \pi c)^{1/2} \) to \( 11(0.4 \pi c)^{1/2} \), and if we assume \( f(x) \) to be constant in this interval of length \( 2(0.4 \pi c)^{1/2} \), we find

\[
g(y) = \frac{1}{2(0.4 \pi c)^{1/2}} \left( \frac{\pi c}{y} \right)^{1/2} = \frac{1}{2(0.4 \pi y)^{1/2}}, \quad 8.1 \leq y \leq 12.1.
\]

The resulting density function \( \phi(\eta) \) is a monotonie decreasing function in the interval from \( \eta = 0.1 \) to \( \eta = 1.1 \). If we divide the whole circle in two parts from \( \eta = 0.1 \) to \( \eta = 0.6 \) and from 0.6 to 1.1, it follows that the probability of a rest position in the first of these semicircles is

\[
\int_{8.1}^{9.1} g(y) dy + \int_{9.1}^{10.1} g(y) dy + \int_{10.1}^{11.1} g(y) dy = 0.506.
\]

The excess of 1.2% is doubtless too large for a fair game of chance. It seems that in such cases other circumstances increase the tendency towards uniformity.

**University of Istanbul**