ON CONTINUED FRACTIONS REPRESENTING CONSTANTS*

H. S. WALL

1. Introduction. Let \( \xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots \) be an infinite sequence of points \( x = (x_1, x_2, x_3, \ldots, x_m) \) in a space \( S \), and let \( \phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_k(x) \) be single-valued real or complex functions over \( S \). Then the functionally periodic continued fraction

\[
1 + \frac{\phi_1(x^{(1)})}{1 + \frac{\phi_2(x^{(1)})}{1 + \cdots + \frac{\phi_k(x^{(1)})}{1 + \frac{\phi_1(x^{(2)})}{1 + \cdots}}}}
\]

is a function \( f(\xi) \) of the sequence \( \xi \). By a neighborhood of a sequence \( \xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots \), we shall understand a set \( N_\xi \) of sequences subject to the following conditions: (i) \( \xi \) is in \( N_\xi \); (ii) if \( \eta: y^{(1)}, y^{(2)}, y^{(3)}, \ldots \) is in \( N_\xi \), then \( \eta_v: y^{(r+1)}, y^{(r+2)}, y^{(r+3)}, \ldots \) and \( \xi_v: y^{(r)} \), \( y^{(s)} \), \( y^{(t)} \), \( y^{(u)} \), \( x^{(r+1)}, x^{(r+2)}, x^{(r+3)}, \ldots \) are in \( N_\xi \) for \( v = 1, 2, 3, \ldots \).

Let \( A_n(\xi) \) and \( B_n(\xi) \) be the numerator and denominator, respectively, of the \( n \)th convergent of \( f(\xi) \) as computed by means of the usual recursion formulas. Put

\[
L(\xi, t) = B_{k-1}(\xi)t^2 + [\phi_k(x^{(1)})B_{k-2}(\xi) - A_{k-1}(\xi)]t - \phi_k(x^{(1)})A_{k-2}(\xi).
\]

Then our principal theorem is as follows:

**Theorem 1.** Let there be a sequence \( c: c^{(1)}, c^{(2)}, c^{(3)}, \ldots \), and a neighborhood \( N_c \) of \( c \), and a number \( r \) having the following properties:

(a) \( f(\xi) \) converges uniformly over \( N_c \),
(b) \( f(c) = r \),
(c) \( L(\xi, r) = 0 \) for every sequence \( \xi \) in \( N_c \),
(d) \( \phi_i(x^{(v)}) \neq 0 \), \( (v = 1, 2, 3, \ldots, i = 1, 2, 3, \ldots, k) \), for every sequence \( \xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots \) in \( N_c \).

When these conditions are fulfilled, \( f(\xi) = r \) throughout \( N_c \).

The proof of Theorem 1 is contained in §2; §3 contains a specialization and §4 an application of this theorem. In §5 continued fractions

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representing constants are obtained by means of certain transformations.*

2. Proof of Theorem 1. Let \( \eta : \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \ldots \) be any sequence in \( N_\epsilon \). Then \( \eta_v : \gamma^{(v+1)}, \gamma^{(v+2)}, \gamma^{(v+3)}, \ldots \) is in \( N_\epsilon \), and \( f(\eta_v), (\nu = 0, 1, 2, \ldots; \eta_0 = \eta) \), converges by (a); and

\[
f(\eta_v) = \frac{A_{k-1}(\eta_v)f(\eta_{v+1}) + A_{k-2}(\eta_v)\phi_k(\gamma^{(v+1)})}{B_{k-1}(\eta_v)f(\eta_{v+1}) + B_{k-2}(\eta_v)\phi_k(\gamma^{(v+1)})},
\]

\[
f(\eta_{v+1}) = -\frac{B_{k-2}(\eta_v)f(\eta_v) - A_{k-2}(\eta_v)}{B_{k-1}(\eta_v)f(\eta_v) - A_{k-1}(\eta_v)} \phi_k(\gamma^{(v+1)}).
\]

The determinant of the matrix

\[
\begin{pmatrix}
A_{k-1}(\eta_v) & A_{k-2}(\eta_v)\phi_k(\gamma^{(v+1)}) \\
B_{k-1}(\eta_v) & B_{k-2}(\eta_v)\phi_k(\gamma^{(v+1)})
\end{pmatrix}
\]

is \( \pm\phi_1(\gamma^{(v+1)})\phi_2(\gamma^{(v+1)}) \cdots \phi_k(\gamma^{(v+1)}) \) and is therefore \( \neq 0 \) by (d). Hence the denominators in (1) cannot vanish, for otherwise the numerators would also vanish, which is impossible. It then follows from (c) that if \( f(\eta_v) = r \) for one value of \( \nu \), then \( f(\eta_v) = r \) for all values of \( \nu (=0, 1, 2, 3, \ldots) \). In particular, if \( \xi_v \) is the sequence \( \gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \ldots, \gamma^{(v)}, c^{(v+1)}, c^{(v+2)}, c^{(v+3)}, \ldots \), then \( f(\xi_v) = r, (\nu = 1, 2, 3, \ldots) \).

Now by (a), for every \( \epsilon > 0 \) there exists a \( K \) such that if \( n > K \), \( p = 1, 2, 3, \ldots \),

\[
\left| \frac{A_{n+p}(\xi_v)}{B_{n+p}(\xi_v)} - \frac{A_n(\xi_v)}{B_n(\xi_v)} \right| < \epsilon
\]

for \( \nu = 1, 2, 3, \ldots \). Choose a fixed \( n > K \), and then choose \( p \) so large that \( A_n(\xi_v)/B_n(\xi_v) = A_n(\eta)/B_n(\eta) \). Then on allowing \( p \) to increase to \( \infty \) in (2) we find that

\[
\left| f(\xi_v) - \frac{A_n(\eta)}{B_n(\eta)} \right| \leq \epsilon \quad \text{or} \quad \left| r - \frac{A_n(\eta)}{B_n(\eta)} \right| \leq \epsilon
\]

if \( n > K \). That is, \( f(\eta) = r \). Since \( \eta \) was any sequence in \( N_\epsilon \) our theorem is proved.

3. Specialization of Theorem 1. Let the sequence \( c \) be such that \( f(c) \) is a periodic continued fraction of period \( k \). Let \( r, s \) be the roots of the quadratic equation \( L(c, t) = 0 \). Then* in order for \( f(c) \) to converge to the value \( r \) the following two conditions are both necessary and sufficient, namely:

(a) \( B_{k-1}(c) \neq 0 \),

(b) \( r = s \) or else
\[
|B_{k-1}(c)r + \phi_h(c^{(1)})B_{k-2}(c)| > |B_{k-1}(c)s + \phi_h(c^{(1)})B_{k-2}(c)| \text{ and } A_{k-1}(c) - sB_k(c) \neq 0, \quad (\lambda = 0, 1, 2, \ldots, k-2).
\]

An important and simple sufficient condition for the uniform convergence of \( f(\xi) \) over \( N_\epsilon \) is that
\[
|\phi_1(\xi^{(i)})| \leq \frac{1}{4}, \quad (i = 1, 2, 3, \ldots, k; \nu = 1, 2, 3, \ldots), \quad \text{for every sequence } \xi: x^{(1)}, x^{(2)}, x^{(3)}, \ldots \text{ in } N_\epsilon.
\]

From these remarks and Theorem 1 we then have this result:

**Theorem 2.** Let there be a sequence \( c \) and a neighborhood \( N_\epsilon \) of \( c \) such that (\( \gamma \)) and conditions (c), (d) of Theorem 1 hold. Then if \( f(c) \) is a periodic continued fraction of period \( k \), we have \( f(\xi) = r \) throughout \( N_\epsilon \).

4. Application in the case where \( \phi_1, \phi_2, \phi_3, \ldots, \phi_k \) are polynomials.

If \( k = 1 \), then \( L(\xi, t) = t^2 - t - \phi_1(x^{(1)}) \), so that in order for (c) of Theorem 1 to hold \( \phi_1 \) must be a constant, and \( f(\xi) \) reduces to an ordinary periodic continued fraction.

Let \( k = 2 \). Then \( L(\xi, t) = t^2 + [\phi_2(x^{(1)}) - \phi_1(x^{(1)}) - 1]t - \phi_2(x^{(1)}) \). We shall suppose that \( \phi_\nu(x) = \phi_\nu(x_1, x_2, x_3, \ldots, x_m), \quad (\nu = 1, 2), \) are polynomials in the real or complex variables \( x_1, x_2, x_3, \ldots, x_m \). Let \( a, b \) be the constant terms, and \( G, H \) the coefficients of \( x_1^m x_2^r \cdots x_m^w \) in \( \phi_1 \) and \( \phi_2 \), respectively. Then (c) of Theorem 1 is equivalent to the relations
\[
(b-a)r - b = r(1-r), \quad (H-G)r - H = 0, \quad \text{all } G, H.
\]

If \( r = 0 \), then \( \phi_2 \equiv 0 \), while if \( r = 1 \), then \( \phi_1 \equiv 0 \). Suppose \( r \neq 0, 1 \). Then if either \( G \) or \( H \) is 0, the other is 0 also, and if \( G = H \), their common value is 0. Hence (c) of Theorem 1 takes the form of the following identity:
\[
(r\phi_1) = (r - 1)(\phi_2 + r), \quad r \neq 0, 1.
\]

On referring to Theorem 2 we now have this result:

**Theorem 3.** Let \( \phi_1(x) \) and \( \phi_2(x) \) be polynomials in the real or complex variables \( x_1, x_2, x_3, \ldots, x_m \) connected by the identity (3) with con-

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† Perron, loc. cit., p. 262.
stant terms \( a \) and \( b \), respectively. Let \( r \), in (3), and \( s \) be the roots of the quadratic equation \( t^2 + (b - a - 1)t - b = 0 \) such that \( r = s \) or else \( r + b > |s + b|, s \neq 1 \). Let \( a, b \) be such that \( |a| < \frac{1}{4}, |b| < \frac{1}{4}, a \neq 0, b \neq 0 \). Then there exists a positive constant \( R \) such that throughout the circle \( |x_i^{(v)}| \leq R, (i = 1, 2, \ldots, m; v = 1, 2, \ldots) \), we have

\[
1 + \frac{\phi_1(x^{(1)})}{1} + \frac{\phi_2(x^{(1)})}{1} + \frac{\phi_1(x^{(2)})}{1} + \frac{\phi_2(x^{(2)})}{1} + \cdots = r,
\]

where \( x^{(v)} = (x_1^{(v)}, x_2^{(v)}, \ldots, x_m^{(v)}) \).

In applying Theorem 2 we have taken \( c^{(v)} = (0, 0, 0, \ldots, 0) \) in the sequence \( c \). It is to be observed that, when this is done and Theorem 2 applies, the value of the continued fraction depends upon only the constant terms of the polynomials \( \phi_1, \phi_2, \phi_3, \ldots, \phi_k \).

5. **Singular continued fractions.** Let \( T \) be a transformation which carries the continued fraction \( f = x_0 + K(x_i/1) \) into another continued fraction \( T f = x_0' + K(x_i'/1) \) in such a way that when either \( f \) or \( T f \) converges the other does also and their values are equal. We shall speak of such a transformation as a proper transformation of \( f \). Suppose moreover that for some positive integer \( n \) the elements \( x_i \) of \( f \) are subject to the condition

\[
x_i = x_i', \quad i = n, n+1, n+2, \ldots.
\]

This gives the following formal relation:

\[
x_0 + \frac{x_1}{1 + \cdots + \frac{x_{n-1}}{g_n}} = x_0' + \frac{x_1'}{1 + \cdots + \frac{x_{n-1}'}{g_n}},
\]

from which one may compute the value of the continued fraction

\[
g_n = 1 + \frac{x_n}{1 + \frac{x_{n+1}}{1 + \cdots}}
\]

when the latter converges.

The procedure outlined above will now be carried out for the following proper transformation:* 

\[
x_0' = x_0 + x_1, \quad x_1' = -x_1, \quad x_2' = (1 + x_2)/x_2; \\
T_2: \quad x'_{2n+1} = x_{2n+1}, \quad x'_{2n+2} = (1 + x_{2n+1})(1 + x_{2n+3})/x_{2n+2},
\]

\[
n = 1, 2, 3, \ldots; x_n \neq 0, -1 \text{ if } n > 0.
\]

In this case the relations (5) are satisfied if and only if

* Leighton and Wall, loc. cit., p. 277.
(6) \[ x_{2i+2}^2 = (1 + x_{2i+1})(1 + x_{2i+3}), \quad i = n, n + 1, n + 2, \ldots, \]
where if \( n = 0 \) the first of these relations is to be replaced by \( x_3^2 = (1 + x_0) \). When \( n = 0 \) we have the relation
\[ x_0 + x_1/g_2 = x_0 + x_1 - x_1/g_2 \]
from which to compute \( g_2 \). It follows that, if \( f \) converges, \( g_2 \) must converge and have the value 2; and if \( g_2 \) converges to a value different from 0, \( f \) must converge and \( g_2 = 2 \). Moreover, it is impossible for \( g_2 \) to have the value \( \infty \), for that would imply that \( f = x_0 \) while \( Tf = x_0 + x_1 \neq f \). If we now write out the continued fraction \( g_2 \) and make a change in notation, the following theorem results.

**Theorem 4.** If \( x_1, x_2, x_3, \ldots \) are arbitrary complex numbers \( \neq 0, -1 \), then the continued fraction
\[
1 + \frac{e_1(1 + x_1)^{1/2}}{1} \frac{x_1}{1 + 1 + \frac{e_2[(1 + x_1)(1 + x_2)]^{1/2}}{1 + 1 + \frac{e_3[(1 + x_2)(1 + x_3)]^{1/2}}{1 + 1 + \ldots}}} \quad e_i = \pm 1,
\]
has one of the values 0 or 2 whenever it converges, and it cannot diverge to \( \infty \).

It is interesting to observe that if \( e_i = +1 \), (7) is the formal expansion of 2 into a continued fraction by means of the identity
\[
1 = \frac{(1 + t)^{1/2}}{t} \cdot \frac{1}{1 + (1 + t)^{1/2}}
\]
As a special case we have the expansion
\[
(1 + N)^{1/2} = 1 + \frac{N}{1 + \frac{N + 1}{1 + \frac{N}{1 + \frac{N + 1}{1 + \ldots}}}},
\]
which is valid if \( N \) is a positive integer.

The transformation \( T_2 \) is one of an infinite group of transformations discussed by the writer* elsewhere in this Bulletin. If one obtains the singular continued fractions corresponding to the case \( m = 3 \) (in the notation of §3, p. 589, of that article), the following three theorems result.

* Wall, loc. cit.
Theorem 5. If the continued fraction

\[
1 - \frac{x_1}{1 - \frac{(x_1^2 - x_1 + 1)}{x_1 \frac{x_2}{1 - \frac{(x_2^2 - x_2 + 1)}{1}}}} - \frac{x_2}{1 - \frac{x_2}{1 - \cdots}},\quad x_n \neq 0, \; x_n^2 - x_n + 1 \neq 0,
\]

converges, its value is \((1 \pm i3^{1/2})/2\).

Theorem 6. If the continued fraction

\[
1 - \frac{e_1}{1 - \frac{x_1}{1 - \frac{(2 - x_1)}{e_2 \frac{x_2}{1 - \frac{(2 - x_2)}{e_3 \frac{x_3}{1 - \cdots}}}}}} - \frac{e_1}{1 - \frac{x_1}{1 - \frac{(2 - x_1)}{e_2 \frac{x_2}{1 - \frac{(2 - x_2)}{e_3 \frac{x_3}{1 - \cdots}}}}}} - \frac{e_1}{1 - \frac{x_1}{1 - \frac{(2 - x_1)}{e_2 \frac{x_2}{1 - \frac{(2 - x_2)}{e_3 \frac{x_3}{1 - \cdots}}}}}} - \frac{e_1}{1 - \frac{x_1}{1 - \frac{(2 - x_1)}{e_2 \frac{x_2}{1 - \frac{(2 - x_2)}{e_3 \frac{x_3}{1 - \cdots}}}}}},
\]

\(e_n = \pm 1, \; x_n \neq 0, \; 2,
\)

converges, its value is 0 or 1.

Theorem 7. If the continued fraction

\[
1 - \frac{x_1}{1 - \frac{(1 - 2x_1)}{x_1 \frac{x_2}{1 - \frac{(1 - 2x_2)}{x_2 \frac{x_3}{1 - \cdots}}}} - \frac{x_1}{1 - \frac{(1 - 2x_1)}{x_1 \frac{x_2}{1 - \frac{(1 - 2x_2)}{x_2 \frac{x_3}{1 - \cdots}}}} - \frac{x_1}{1 - \frac{(1 - 2x_1)}{x_1 \frac{x_2}{1 - \frac{(1 - 2x_2)}{x_2 \frac{x_3}{1 - \cdots}}}}}},
\]

\(x_n \neq 0, \; \frac{1}{2},
\)

converges, its value is 0 or \(\frac{1}{2}\).

The proofs of these theorems are along the lines of the proof of Theorem 4, and will be omitted.