and as above this is sufficient that postulates I–V be satisfied.

With this definition of $A:B$, $A \triangleright B$ becomes the usual inclusion relation of the algebra of classes [5].

REFERENCES


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A NOTE ON THE MAXIMUM PRINCIPLE FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Let $u(x_1, \ldots, x_n)$ denote a twice continuously differentiable function of $x_1, \ldots, x_n$ in some region $R$. We write $\partial u/\partial x_i = u_i, \partial^2 u/\partial x_i \partial x_k = u_{ik}$, and occasionally $(x)$ for $(x_1, \ldots, x_n)$. A point $(c) = (c_1, \ldots, c_n)$ of $R$ may be called a proper maximum of $u$, if

\[ u_i(c) = 0 \quad \text{for} \quad i = 1, \ldots, n, \]
\[ \sum_{i,k} u_{ik}(c) \xi_i \xi_k < 0 \quad \text{for all} \quad (\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0). \]

A partial differential equation

(1) \[ \sum_{i,k} a_{ik}(x) u_{ik}(x) + \sum_i b_i(x) u_i(x) = 0 \]

(where the $a_{ik}$ and $b_i$ are defined in $R$) is called elliptic if for every $(x)$ of $R$

\[ \sum_{i,k} a_{ik}(x) \xi_i \xi_k \geq 0 \]

for all $(\xi_1, \ldots, \xi_n)$ and $< 0$ for some $(\xi_1, \ldots, \xi_n)$. 
It is well known, that a solution \( u \) of (1) can not have a proper maximum.\(^*\) For if \( u \) had a proper maximum at \( (c_1, \ldots, c_n) \), then 
\[
\sum_{i,k} a_{ik}(c)u_{ik}(c) = 0.
\]
If \( A \) and \( U \) denote respectively the matrices \( (a_{ik}(c)) \) and \( (u_{ik}(c)) \), this may be written: Trace \( (A \cdot U) = 0 \). By a suitable orthogonal transformation \( A \) may be transformed into a diagonal matrix \( A' = (a'_{i\delta_{ik}}) \), \( U \) going over into \( U' = (u'_{ik}) \) by the same transformation. As the trace of \( A \cdot U \) is preserved, we have \( \sum a'_{i\delta_{ik}}u'_{ik} = 0 \); on the other hand, as \( A' \) still belongs to a semi-definite quadratic form and \( U' \) to a negative definite one, we have \( a'_{i\delta_{ik}} \geq 0 \) for \( i = 1, \ldots, n \), but \( < 0 \) for some \( i \), and \( u'_{ik} < 0 \) for all \( i \). This leads to a contradiction.

A second important property of the solutions \( u \) of (1) is that they form a module, that is, that every linear combination with constant coefficients is again a solution.

We shall prove that these two properties are also sufficient to characterize a family of functions as solutions of an elliptic equation (1).

**Theorem.** Let \( F \) be any family of twice continuously differentiable functions \( u(x_1, \ldots, x_n) \) defined in \( \mathbb{R} \), such that
\begin{enumerate}
  \item the functions of \( F \) form a module,
  \item no function of \( F \) has a proper maximum.
\end{enumerate}

Then there is an elliptic differential equation (1) satisfied by all functions of \( F \).

**Proof.** Let \( (c) = (c_1, \ldots, c_n) \) be a point of \( \mathbb{R} \). Let \( \phi \) be the submodule of functions \( u \) of \( F \) for which \( u_i(c) = 0 \) for \( i = 1, \ldots, n \). Let \( Q(\xi_1, \ldots, \xi_n) \) denote the quadratic form \( \sum_{i,k} u_{ik}(c)\xi_i\xi_k \) for any \( u \) in \( \phi \); \( Q \) is certainly not negative definite. These quadratic forms form again a module \( M \). Let \( Q_1, Q_2, \ldots, Q_m \) form a basis of this module \( (m \leq n(n+1)/2) \), such that for every \( Q \) of \( M \)
\[
Q(\xi_1, \ldots, \xi_n) = \sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \ldots, \xi_n)
\]
with certain constants \( \lambda_i \). We know that for every \( (\lambda_1, \ldots, \lambda_m) \) there are \( (\xi_1, \ldots, \xi_n) \neq (0, \ldots, 0) \) such that
\[
\sum_{i=1}^{m} \lambda_i Q_i(\xi_1, \ldots, \xi_n) \geq 0.
\]
From this we can easily conclude that the \( Q_i \) satisfy a linear relation

with positive coefficients.* For let \( \Sigma \) denote the set of points with coordinates \((Q_1(\xi), Q_2(\xi), \ldots, Q_m(\xi))\) in an \( m \)-dimensional space, where \((\xi) = (\xi_1, \ldots, \xi_n)\) varies over all points of the unit sphere \(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = 1\). The set \( \Sigma \) is closed and finite. The relation (2) may be interpreted as stating that every half-space bounded by a plane through the origin contains points of \( \Sigma \). Thus the origin is contained in the convex extension of \( \Sigma \). Then there exists a finite set of points \( P_1, \ldots, P_r \) of \( \Sigma \) and positive numbers \( \mu_1, \ldots, \mu_r \), such that the origin is the center of mass of the masses \( \mu_i \) placed at the vertices \( P_i \).† Let \((\xi_1^i, \ldots, \xi_n^i)\) be the point \((\xi_1, \ldots, \xi_n)\) corresponding to \( P_i \). Then

\[
\sum_{j=1}^r \mu_j Q_k(\xi_1^j, \ldots, \xi_n^j) = 0
\]

for \( k = 1, \ldots, m \), and consequently

\[
\sum_{j=1}^r \mu_j Q(\xi_1^j, \ldots, \xi_n^j) = 0
\]

for every \( Q \) in \( M \). Thus

\[
\sum_{j=1}^r \sum_{i,k} \mu_j \delta_{ik} \xi_1^j \xi_k = 0
\]

for \( u \) in \( \phi \). Putting \( \sum \mu_j \delta_{ik} \xi_1^j \xi_k = a_{ik}(c) \), we have

\[
\sum_{i,k} a_{ik}(c) u_{ik}(c) = 0
\]

for \( u \) in \( \phi \). Besides

\[
\sum_{i,k} a_{ik}(c) \xi_k \xi_k = \sum_{i,j} \mu_j (\xi_1^i \xi_1^j)^2 \geq 0
\]

and \( > 0 \) for some \((\xi_1, \ldots, \xi_n)\).

Now let \( u(x_1, \ldots, x_n) \) denote an arbitrary function of \( F \). The vectors \((y_1, \ldots, y_n) = (u_1(c), u_2(c), \ldots, u_n(c))\) again form a module \( N \), if \( u \) varies over \( F \) for fixed \((c)\). Let \((y_1^1, y_1^2, \ldots, y_1^n), (y_2^1, \ldots, y_2^n), \ldots, (y_s^1, \ldots, y_s^n)\) form a basis of \( N \), \((s \leq n)\). Without restriction of generality we may assume that this basis forms an orthogonal system:

\[
\sum_{l=1}^n y_l^i y_l^k = \delta_{ik}, \quad i, k = 1, \ldots, s.
\]


† Bonnesen-Fenchel, Theorie der Konvexen Körper, p. 9.
Let \( u^1(x, c), u^2(x, c), \ldots, u^s(x, c) \) be the functions of \( F \) corresponding to the vectors
\[
u_k^i(x, c, c) = y^k_i.
\]
Then for every \( u \) in \( F \)
\[
u_i(c) = \sum_{j=1}^{s} \lambda_j y^j_i
\]
with
\[
\lambda_j = \sum_{k=1}^{n} u_k(c) y^j_k.
\]
Thus
\[
u_i(c) = \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u^j_i(c, c).
\]
Consider the function
\[
\tilde{u}(x) = u(x) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u^j_i(x, c).
\]
Then \( \tilde{u} \) is in \( F \). We have
\[
\tilde{u}_i(c) = u_i(c) - \sum_{j=1}^{s} \sum_{k=1}^{n} u_k(c) u_k^j(c, c) u^j_i(c, c) = 0.
\]
Hence \( \tilde{u} \) is in \( \phi \). Consequently
\[
0 = \sum_{i, k} a_{ih}(c) \tilde{u}_{ih}(c) = \sum_{i, h} a_{ih}(c) u_{ih}(c) + \sum_{i, j, h} a_{ih}(c) u_k(c) u_k^j(c, c) u^j_i(c, c).
\]
Thus \( u(x) \) satisfies the elliptic equation
\[
0 = \sum_{i, h} a_{ih}(x) u_{ih}(x) - \sum_{i, j, h} a_{ih}(x) u_k^j(x, x) u^j_i(x, x) u_k(x).
\]