ON THE $n$TH DERIVATIVE OF $f(x)^*$

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Let $y_1$, $y_2$, $y_3$, \ldots be defined recursively as follows: $y_1$ is the logarithmic derivative of a function $y = f(x)$, and $y_v = D_x y_{v-1}$, ($v = 2$, $3$, $4$, \ldots). Then the successive derivatives $y'$, $y''$, $y'''$, \ldots of $y$ with respect to $x$ are polynomials in $y$ and the $y_v$. In fact, $y' = y y_1$, $y'' = y (y_2 + y_1^2)$, $y''' = y (y_3 + 3 y_1 y_2 + y_1^3)$, and

\[
y^{(n)} = y \sum A_{n_1 n_2 \ldots n_n}^{(n)} y_1^{n_1} y_2^{n_2} \cdots y_n^{n_n},
\]

where $A_{n_1 n_2 \ldots n_n}^{(n)}$ is a positive integer and the summation is taken for all non-negative integral solutions $n_1, n_2, n_3, \ldots, n_n$ of the equation

\[
n_1 + 2n_2 + 3n_3 + \cdots + nn_n = n.
\]

This statement may readily be proved by mathematical induction. The principal object of the present note is to prove the following theorem:

**Theorem.** The integer $A_{n_1 n_2 \ldots n_n}^{(n)}$ in (1) is equal to the number of ways that $n$ different objects can be placed in compartments, one in each of $n_1$ compartments, two in each of $n_2$ compartments, three in each of $n_3$ compartments, $\ldots$, without regard to the order of arrangement of the compartments.

1. Generalized binomial coefficients. Let $k$, $m$, $n$, $(kn \leq m)$, be positive integers, and denote by $C_{m,n}^{(k)}$ the number of ways that $kn$ objects can be selected from $m$ objects and placed in $n$ compartments, $k$ in each compartment, where no account is taken of the order of arrangement of the compartments. Thus $C_{m,n}^{(k)}$ is the binomial coefficient $C_{m,n}^{(k)} = \binom{m}{kn} \cdot \binom{kn}{k} \cdot \cdots \cdot \binom{k}{k}$.

We have

\[
n! \cdot C_{m,n}^{(k)} = C_{m, kn} \cdot (C_{kn,k} \cdot C_{k(n-1),k} \cdot \cdots \cdot C_{k,k}),
\]

or

\[
C_{m,n}^{(k)} = m! \left( \frac{1}{[n!(m - kn)!(k!)^n]} \right).
\]

This has meaning if $m \geq kn$. For special 0 values of the indices we shall consider $C_{m,n}^{(k)}$ to be defined by (3) by taking $0! = 1$. Thus if $k \geq 0$, $m \geq 0$, we have $C_{m,0}^{(k)} = 1$.

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If (2) holds, it will be seen that

\[ C_{k_1,\nu_1}^{(1)} \cdot C_{k_2,\nu_2}^{(2)} \cdots C_{k_n,\nu_n}^{(n)} = \frac{n!}{\nu_1! \nu_2! \cdots \nu_n! (1!)^{\nu_1} (2!)^{\nu_2} \cdots (n!)^{\nu_n}}, \]

where \( k_1 = n, \ k_2 = n - \nu_1, \ k_3 = n - \nu_1 - 2\nu_2, \ \cdots \). We are to prove that this is the value of \( A_{\nu_2,\ldots,\nu_n}^{(n)} \) in (1). For the proof we need the following identities which will be seen to hold for all values of \( m, n, k \) for which the symbols involved have been defined:

\[ C_{m,n}^{(k+1)} = C_{m-1,n}^{(k)} + C_{m-1,1}^{(k)} \cdot C_{m-k-1,n-1,1}^{(k+1)}, \]

\[ (n + 1) \cdot C_{m,n+1}^{(k)} = C_{m,n}^{(k)} \cdot C_{m-k,n+1}^{(k)} \cdot C_{m-k,n+1}^{(k+1)}. \]

Let it be remarked in passing that if \( P_{n,k} = C_{n,0}^{(k)} + C_{n,1}^{(k)} x + C_{n,2}^{(k)} x^2 + \cdots \), then from (5) it follows that \( P_{n,k} = P_{n-1,k} + C_{n-1,k-1} x P_{n-k,k} \).

Also \( P_{n,k} = C_{n,k} P_{n-k,k} \).

2. Derivation of the formula for \( A_{\nu_2,\ldots,\nu_n}^{(n)} \). Denote the sum in (1) by \( S_n \), and write \( S_n \) as a polynomial in \( y_1 \), \( S_n = \sum_{\nu=0}^n S_n^{(1)} y_1^\nu \), where \( S_n^{(1)} \) is independent of \( y_1 \). We begin by showing that

\[ S_n^{(1)} = S_n^{(1)} + C_{i,n-j}^{(1)} S_n^{(1)}, \quad 0 \leq j \leq i. \]

We use induction on the subscript difference \( i-j = k \). From the relation \( yS_n = D_x [y S_{n-1}] \) it follows that

\[ S_{n,v} = S_{n-1,v-1}^{(1)} + (1 + v) y_2 S_{n-1,v+1}^{(1)} + D_x S_{n-1,v}^{(1)}, \]

\[ \nu = 0, 1, 2, \ldots, n, \]

with the agreement that \( S_{i,j} = 0 \) if \( j < 0 \) or \( j > i \). Assuming that (7) holds for \( k < q \) we shall prove that it holds for \( k = q \). Accordingly, we choose \( n, \nu, (0 \leq \nu < n) \), in (8) so that \( n - \nu = q \). Then, by our assumption, (8) may be written in the form

\[ S_{n,v}^{(1)} = \begin{cases} S_{n-1,v-1}^{(1)} + (1 + v) y_2 C_{n-1,v+1}^{(1)} S_{n,v-2,0}^{(1)} + C_{n-1,v} D_x S_{n-1,v-1,0}^{(1)}, & \text{if } q > 1; \\
S_{n-1,v-1}^{(1)} + C_{n-1,v} D_x S_{n-1,v-1,0}^{(1)}, & \text{if } q = 1. \end{cases} \]

Replace \( n \) by \( n-\nu \), and \( v \) by 0 in (8), and eliminate \( D_x S_{n-\nu-1,0}^{(1)} \) in (9). The result, by (6), is

\[ S_{n,v}^{(1)} = S_{n-1,v-1}^{(1)} + C_{n-1,v} S_{n-\nu,0}^{(1)}. \]
Hence
\[ S^{(1)}_{n,v} = \sum_{i=0}^{v} \left[ S^{(1)}_{n-i,v-i} - S^{(1)}_{n-i-1,v-i-1} \right] = \left[ \sum_{i=0}^{v} C^{(1)}_{n-i-1,v-i} \right] S^{(1)}_{n,v,0}; \]
or, by (5) with \( k = 0 \), \( S^{(1)}_{n,v} = C^{(1)}_{n,v} S_{n,v,0} \), as was to be proved.

We next put
\[ S^{(p-1)}_{m,0} = \sum_{\nu=0}^{[m/p]} S^{(p)}_{m,v} y^{p}_{\nu}, \quad p = 2, 3, 4, \cdots. \]

Then the formulas
(11) \[ S^{(p)}_{m,v} = C^{(p)}_{m,v} S^{(p)}_{m-pr,0}, \]
(12) \[ S^{(p)}_{m,v} = C^{(p-1)}_{m-1,1} S^{(p)}_{m-p,v-1} + (1 + \nu) y^{p+1}_{\nu+1} S^{(p)}_{m-1,v+1} + D_{\nu} S^{(p)}_{m,v}, \]
hold for \( p = 1 \). Assuming that they hold for \( p < k \), \( k > 1 \), we may then prove them for \( p = k \). To do this, put \( \nu = 0 \) and \( p = k - 1 \) in (12), and equate coefficients of like powers of \( y_k \). The result is the equation (12) with \( p = k \). Thus (12) is true when \( p = k \), and in particular \( S^{(k)}_{kr,v} = C^{(k-1)}_{kr-1,1} S^{(k)}_{kr-1,v-1} \). Hence by (5) we find that \( S^{(k)}_{kr,v} = C^{(k-1)}_{kr,v} S^{(k)}_{kr-1,v} \), so that (11) holds for \( p = k \) provided \( m - kv = 0 \). The proof of (11) for \( p = k \) may now be carried out along the lines of the proof of (7), with induction, in this case, on the difference \( m - kv \).

After (11) has been proved it follows at once by (4) that
\[ A^{(n)}_{\nu_1 \nu_2 \cdots \nu_n} = \frac{n!}{\nu_1! \nu_2! \cdots \nu_n! (1)! \cdots (n-1)! \cdots (n)!} \cdot \]

3. Application. In conclusion I shall give examples to illustrate the application of the foregoing result.

Example 1. Let \( a_{n,k} \) denote the number of ways that \( n \) different objects can be distributed among \( k \) compartments, where no account is taken of the order of arrangement of the compartments, and at least one object is placed in each compartment. Then elementary considerations will show that \( a_{n,k} = k a_{n-1,k} + a_{n-1,k-1} \). Put
\[ y = e^{t \nu_1} = e^{\nu_1 \sum_{\nu=0}^{\infty} L_{\nu}(t) x^{\nu}/\nu!}. \]

Then by (1) we find that \( L_{\nu}(t) = a_{\nu,1} t + a_{\nu,2} t^2 + \cdots + a_{\nu,n} t^n \).

Put \( g_k(y) = \sum_{\nu=0}^{\infty} a_{k+\nu,y} y^\nu \). It follows that \( g_1(y) = 1/(1-y) \), \( g_k(y) = g_{k-1}(y)/(1-ky) \), \( (k = 2, 3, 4, \cdots) \), or \( g_k(y) = 1/[1-y(1-2y) \cdots (1-ky)] \). Hence \( L_n(1) \), the number of ways that \( n \) different ob-
jects can be distributed among \( n \) or fewer compartments, is the coefficient of \( y^n \) in the power series \( P(y) \) for the function

\[
\frac{y}{1 - y} + \frac{y^2}{(1 - y)(1 - 2y)} + \cdots + \frac{y^m}{(1 - y)(1 - 2y) \cdots (1 - my)},
\]

where \( m \geq n \). The number \( L_n(1) \) is also the coefficient of \( x^n/n! \) in the power series for the function \( e^{(e^x-1)} \).

The \( a_{n,k} \) are given explicitly by the formula

\[
a_{n,k} = \frac{(-1)^{k+1}}{k!} \sum_{r=0}^{k} C_{k,r} (-1)^{r+1} r^n.
\]

**Example 2.** Put \( y = (1 + x)^{-t} \) in (1) and then set \( x = 0 \). There results this identity:

\[
t(t + 1)(t + 2) \cdots (t + n - 1)
\]

\[
= \sum \frac{t^{v_1+v_2+\cdots+v_n}}{(1 \cdot 2 \cdot \cdots \cdot v_1)(2 \cdot 4 \cdot \cdots \cdot 2v_2) \cdots (n \cdot 2n \cdot \cdots \cdot n v_n)},
\]

where the summation is taken as in (1). On putting \( t = 1 \) in (14) we obtain the following theorem.*

**Theorem.** Form a partition of \( n \) by taking at most one integer from each of the progressions 1, 2, 3, \cdots ; 2, 4, 6, \cdots ; 3, 6, 9, \cdots ; \cdots . Multiply together the terms of each progression up to and including the integer chosen. Let the products so formed be \( a, b, c, \cdots \). Then

\[
\sum [1/(a \cdot b \cdot c \cdots )] = 1,
\]

where the sum is taken for all such partitions of \( n \).

**Example 3.** If we differentiate the members of (14) with respect to \( t \) and then set \( t = 1 \), we get the formula

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]

\[
= \sum \frac{(v_1 + v_2 + \cdots + v_n)}{(1 \cdot 2 \cdot \cdots \cdot v_1)(2 \cdot 4 \cdot \cdots \cdot 2v_2) \cdots (n \cdot 2n \cdot \cdots \cdot n v_n)}.
\]

This may likewise be interpreted as a theorem on partitions of \( n \).

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