RELATIONS AMONG THE FUNDAMENTAL SOLUTIONS
OF THE GENERALIZED HYPERGEOMETRIC
EQUATION WHEN \( p = q + 1 \).

I. NON-LOGARITHMIC CASES*

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It is the purpose of this paper to develop the relations among the non-logarithmic solutions of the equation

\[
\left\{ \prod_{t=1}^{p} (\theta + a_t) - \frac{1}{z} \prod_{t=1}^{q} (\theta + c_t - 1) \right\} y = 0, \quad p = q + 1,
\]

where \( \theta = z(d/dz) \). The results of this paper generalize those of Mehlenbacher,† who studied the case in which \( p - 2 \) and \( q = 1 \), and extend those of Barnes‡ who obtained the asymptotic developments of the non-logarithmic solutions of (1) for those cases in which \( p < q + 1 \).

The succeeding analysis will be simplified if we rewrite equation (1) in the equivalent form

\[
\left\{ \prod_{t=1}^{q+1} (\theta + a_t) - \frac{1}{z} \prod_{t=1}^{q+1} (\theta + c_t - 1) \right\} y = 0, \quad c_{q+1} = 1.
\]

If no two of the \( a_t \) or \( c_t \) are equal or differ by an integer, then equation (2) has \( q + 1 \) linearly independent solutions about the point \( z = 0 \), which may be written

\[
Y_{0j} = z^{-c_j} \prod_{t=1}^{q+1} \frac{\Gamma(1 + c_t - c_j)}{\Gamma(1 + a_t - c_j)} \sum_{n=0}^{\infty} \frac{\Gamma(1 + a_t - c_j + n)}{\Gamma(1 + c_t - c_j + n)} z^n,
\]

\( j = 1, 2, 3, \ldots, q + 1; c_{q+1} = 1; \quad |z| < 1; \)

and \( q + 1 \) linearly independent solutions about the point \( z = \infty \) which may be written

\[
Y_{\infty j} = z^{-a_j} \prod_{t=1}^{q+1} \frac{\Gamma(1 - a_t + a_j)}{\Gamma(1 - c_t + a_j)} \sum_{n=0}^{\infty} \frac{\Gamma(1 - c_t + a_j + n)}{\Gamma(1 - a_t + a_j + n)} z^n,
\]

\( j = 1, 2, 3, \ldots, q + 1; c_{q+1} = 1; \quad |z| > 1. \)

* Presented to the Society, December 30, 1937. The logarithmic cases will be considered in a later paper.
† See this Bulletin, Abstract 42-5-169.
In this paper, we shall develop the relations among solutions (3) and (4) of equation (2).

In this connection, we may state the following theorem:

**Theorem 1.** If no two of the \( a_t \) or \( c_t \) are equal or differ by an integer, then the solutions (3) of equation (2), when extended analytically outside their circle of convergence, may be expressed linearly in terms of solutions (4) in the following forms, it being understood throughout that \( 0 < \arg z < 2\pi \):

\[
Y_{0j} = \sum_{k=1}^{q+1} \left\{ e^{i\pi(a_k-c_{j+1})} \frac{\Gamma(c_j-a_k)\Gamma(1+c_k-c_j)}{\Gamma(c_k-a_k)} \prod_{l=1 \atop l \neq k}^{q+1} \frac{\Gamma(a_l-a_k)\Gamma(1+c_l-c_j)}{\Gamma(c_l-a_k)\Gamma(1+a_l-c_j)} Y_{\infty l} \right\} ,
\]

\[j = 1, 2, 3, \ldots, q + 1; c_{q+1} = 1.\]

**Proof.** It is well known that the function

\[
g(w) = \prod_{l=1}^{q+1} \frac{\Gamma(1+a_l-c_l+w)}{\Gamma(1+c_l-c_l+w)}
\]

satisfies the relation

\[
g(w) = \sum_{l=1}^{q+1} (a_l-c_l) [1 + \gamma(w)] \frac{\Gamma(1+a_l-c_l+w)}{\Gamma(1+c_l-c_l+w)}, \quad \lim_{|w| \to \infty} \gamma(w) = 0, \quad |\arg w| < \pi. \tag{6}
\]

Therefore,

\[
|g(w)| \leq |w|^{\sum_{l=1}^{q+1}(a_l-c_l)} |1 + |\gamma(w)|| \leq |w| R e^{-I\phi} |1 + |\gamma(w)|| ,
\]

where \( R \) and \( I \) denote the real and imaginary parts of \( \sum_{l=1}^{q+1}(a_l-c_l) \), respectively, and where \( \phi = \arg w \). For sufficiently large values of \( |w| \), we have \( |\gamma(w)| < \epsilon, \quad |\arg w| < \pi. \) Thus, for such values of \( |w| \)

\[
|g(w)| < |w| R e^{-I\phi}(1 + \epsilon) < |w| R e^{\frac{\pi}{2}}(1 + \epsilon) < K |w| R ,
\]

where \( K \) is a constant independent of \( w \). The function \( g(w) \), therefore, satisfies the conditions of a general theorem due to Ford, \( \dagger \) but, as ap-

\* As far as relation (5) itself is concerned, the restriction \( c_{q+1} = 1 \) is unnecessary. The restriction is necessary, however, in order that the \( Y_{0j} \) and the \( Y_{\infty l} \) be solutions of equation (2).

\( \dagger \) See, for example, Milne-Thomson, The Calculus of Finite Differences, Macmillan, 1933, p. 255.

pears from (6), has simple poles at the points

(7) \( w = c_j - a_k - n - 1, \quad n = 0, 1, 2, \ldots; k = 1, 2, 3, \ldots, q + 1. \)

An application of this theorem to the function

(8) \[ f(z) = \sum_{n=0}^{\infty} g(n)z^n = \sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + n)}{\Gamma(1 + c_t - c_j + n)} z^n \]

gives the result

(9) \[ f(z) \sim -\sum_{n=0}^{\infty} \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j - n)}{\Gamma(1 + c_t - c_j - n)} \frac{1}{z^n} - \sum_{k=1}^{q+1} \sum_{n=0}^{\infty} R_{n,k}, \]

where \( R_{n,k} \) represents the residue of the function

(10) \[ \frac{\pi(-z)^w g(w)}{\sin \pi w} = (-z)^w \Gamma(w) \Gamma(1 - w) \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} \]

at the point \( w = c_j - a_k - n - 1 \). But the first series on the right in (9) drops out since the denominator of every term contains a factor for which \( t = j \), and

\[ \frac{1}{\Gamma(1 + c_j - c_j - n)} = \frac{1}{\Gamma(1 - n)} = 0, \quad n = 1, 2, 3, \ldots. \]

It follows that our problem reduces to that of computing the residue term.

For this purpose, consider

(11) \[ R_{n,k} = \frac{1}{2\pi i} \int_{C_{n,k}} (-z)^w \Gamma(w) \Gamma(1 - w) \prod_{t=1}^{q+1} \frac{\Gamma(1 + a_t - c_j + w)}{\Gamma(1 + c_t - c_j + w)} \, dw, \]

where \( C_{n,k} \) surrounds the pole \( w = c_j - a_k - n - 1 \) of (10) but no other pole of (10). If in (11) we replace \( w \) by \( -(w - c_j + a_k + n + 1) \), we obtain

\[ R_{n,k} = \frac{(-z)^{c_j - a_k - n - 1}}{2\pi i} \]

(12) \[ \int_{C_{\lambda}} (-z)^{-w} \frac{\Gamma(c_j - a_k - w - n - 1) \Gamma(w - c_j + a_k + n + 2) \Gamma(-w - n)}{\Gamma(c_k - a_k - w - n)} \prod_{t=1}^{q+1} \frac{\Gamma(a_t - a_k - w - n)}{\Gamma(c_t - a_k - w - n)} \, dw, \]
where $C_0$ surrounds the origin. By several applications of the relation

$$\Gamma(1 - w) = \frac{\pi}{\Gamma(w) \sin \pi w},$$

we may change (12) to the form

$$R_{n,k} = -\frac{(-z)^{c_j - a_k - n - 1}(-1)^n}{2\pi i} \int_{C_0} \frac{\pi(-z)^{-w} \Gamma(c_j - a_k - w)\Gamma(w - c_j + a_k + 1)}{\sin \pi w \Gamma(c_j - a_k - w)\Gamma(w - c_j + a_k + 1)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k - w)\Gamma(w - a_t + a_k + 1)}{\Gamma(c_t - a_k - w)\Gamma(w - c_t + a_k + 1)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(w - c_t + a_k + n + 1)}{\Gamma(w - c_t + a_k + n + 1)} \, dw,$$

which by virtue of a well known theorem in the calculus of residues* is equal to

$$R_{n,k} = -\frac{(-z)^{c_j - a_k - 1} \Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{z^n \Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 - a_t + a_k)}{\Gamma(c_t - a_k)\Gamma(1 - c_t + a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(1 - c_t + a_k + n)}{\Gamma(1 - a_t + a_k + n)}.$$

Therefore,

$$\sum_{n=0}^{\infty} R_{n,k} = -e^{i\pi(a_k - c_j + \frac{1}{2})} (-z)^{c_j - a_k - 1} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k)\Gamma(1 - a_t + a_k)}{\Gamma(c_t - a_k)\Gamma(1 - c_t + a_k)} \cdot \sum_{n=0}^{\infty} \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(1 - c_t + a_k + n)1}{\Gamma(1 - a_t + a_k + n)z^n}$$

$$= -e^{i\pi(a_k - c_j + \frac{1}{2})} (-z)^{c_j - a_k - 1} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_j - a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k)}{\Gamma(c_t - a_k)} \cdot Y_{\infty,k}.$$

Thus,

* See Ford, op. cit., p. 5, equation (20).
Substituting (17) into (9) and recalling that the first series of (9) is no longer present, we have

\[ f(z) \sim z^{c_j - 1} \sum_{k=1}^{q+1} \left\{ e^{i\pi (a_k - c_j + c_k)} \frac{\Gamma(c_j - a_k)\Gamma(1 - c_j + a_k)}{\Gamma(c_k - a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(a_t - a_k)}{\Gamma(c_t - a_k)} Y_{\omega_k} \right\}. \]  

Since each \( Y_{\omega_k} \) which appears in (18) is defined by a series which converges for \(|z| > 1\), we may replace \( \sim \) by = in (18). Then, upon noting that

\[ Y_{0j} = z^{1-c_j} \prod_{t=1}^{q+1} \frac{\Gamma(1 + c_t - c_j)}{\Gamma(1 + a_t - c_j)} f(z), \]  

we see that (18) is equivalent to the desired result (5).

By means of a similar proof, we may establish the following theorem:

**Theorem 2.** If no two of the \( a_t \) or \( c_t \) are equal or differ by an integer, then the solutions (4) of equation (2), when extended analytically outside their circle of convergence, may be expressed linearly in terms of solutions (3) in the following forms, it being understood throughout that \( 0 < \text{arg } (1/z) < 2\pi \):

\[ Y_{\omega_j} = \sum_{k=1}^{q+1} \left\{ e^{i\pi (a_j - c_k + 1)} \frac{\Gamma(c_k - a_j)\Gamma(1 - a_k + a_j)}{\Gamma(c_k - a_k)} \cdot \prod_{\substack{t=1 \atop t \neq k}}^{q+1} \frac{\Gamma(c_k - c_t)\Gamma(1 - a_t + a_j)}{\Gamma(c_k - a_t)\Gamma(1 - c_t + a_j)} Y_{\omega_k} \right\}, \]

\[ j = 1, 2, 3, \ldots, q + 1; c_{q+1} = 1. \]