A NOTE ON FREDHOLM-STIELTJES INTEGRAL EQUATIONS*

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1. Introduction. The object of this paper is to show that the integral equation†

\[ f(x) = m(x) + \lambda \int_0^1 f(y)dG(x, y), \quad 0 \leq x, y \leq 1, \]

can be changed into an ordinary Fredholm equation when \( G(x, y) \) is absolutely continuous \( g(y) \).‡ The integration is carried out in the Young-Stieltjes sense, and \( g(y) \) is a bounded, monotone increasing function.

2. Lemmas. If \( h(x) \) is of bounded variation and we set \( h(x) = h(0), \ (x < 0), \) and \( h(x) = h(1), \ (x > 1), \) then we may define the completely additive function of sets \( \tilde{h}(e) \) by

\[ \tilde{h}(e) = h(x_2 + 0) - h(x_1 - 0), \quad e = e(x_1 \leq t \leq x_2). \]

Using this notation we have the following lemma:

**Lemma 1.** If \( f(x) \) is measurable Borel then

\[ \int_0^1 f(x)dh(x) = \int_0^1 f(x)d\tilde{h}, \]

the left side being Young-Stieltjes integration, the right Radon-Stieltjes.

In case one integral does not exist the equality sign is taken to mean that the other integration is non-existent. Because of the properties of the integrals under consideration, we need only prove the equality for the functions

\[ f_1(x) = 1, \ x = \alpha, \quad f_2(x) = 1, \ 0 \leq \alpha < x < \beta \leq 1, \]

\[ = 0, \ x \neq \alpha; \quad = 0, \text{ elsewhere.} \]

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* Presented to the Society, December 29, 1936.

All functions used in the present paper are assumed to be measurable Borel.
We have
\[ \int_0^1 f_1(x) d\bar{h}(x) = h(\alpha + 0) - h(\alpha - 0) = \bar{h}(\alpha) = \int_0^1 f_1(x) d\bar{h}, \]
\[ \int_0^1 f_2(x) d\bar{h}(x) = h(\beta - 0) - h(\alpha + 0) = \bar{h}(\epsilon) = \int_0^1 f_2(x) d\bar{h}, \]
where \( \epsilon \) is the open set \( \alpha < t < \beta. \)

**Lemma 2.** If \( G(x) \) is absolutely continuous with respect to the bounded monotone increasing function \( g(x) \), then
\[ \int_0^1 f(x) dG(x) = \int_0^1 f(x) DG(x) dg(x), \]
where \( DG(x) \) is the derivative or one of the derived numbers of \( G(x) \) with respect to \( g(x) \).

Mr. Maria† has made the important step in the proof of the lemma by showing that
\[ G(x_2 + 0) - G(x_1 - 0) = \int_E DG(x) d\bar{g}, \]
where \( E \) is the set \( x_1 \leq t \leq x_2 \). For the function \( f_1(x) \), making use of Lemma 1, we have
\[ \int_0^1 f_1(x) dG(x) = G(\alpha + 0) - G(\alpha - 0), \]
\[ \int_0^1 f_1(x) DG(x) dg(x) = \int_0^1 f_1(x) DG(x) d\bar{g} = \int_E DG(x) d\bar{g} 
= G(\alpha + 0) - G(\alpha - 0), \]
where \( E \) is the point \( \alpha \). For \( f_2(x) \) we have, if \( \epsilon \) is the open set \( \alpha < x < \beta, \)
\[ \int_0^1 f_2(x) dG(x) = G(\beta - 0) - G(\alpha + 0), \]
\[ \int_0^1 f_2(x) DG(x) dg(x) = \int_0^1 f_2(x) DG(x) d\bar{g} = \int_\epsilon DG(x) d\bar{g} 
= G(\beta - 0) - G(\alpha + 0). \]

From the above material the lemma readily follows.

* The same reasoning shows that \( \int_0^1 f(t) dh(t) \) is equal to \( \int_0^1 f(t) d\bar{h}, \) for \( 0 < x < 1, \) if \( h(t) \) is continuous from the right except perhaps at \( x = 0. \)
† Loc. cit., p. 430.
3. Transformations. Our first theorem is the following.

**Theorem 1.** If $G(x, y)$ is absolutely continuous $g(y)$ then equation (1) can be written in the form

$$f(x) = m(x) + \lambda \int_0^1 K(x, y)f(y)dg(y),$$

where $K(x, y) = DG(x, y)$, the derivative being taken with respect to $g(y)$, a bounded monotone increasing function.

This is immediate from Lemma 2.

**Theorem 2.** If $m(x)$ and $K(x, y)$ are bounded, then the solution of (1) and (2), except for characteristic values of $\lambda$, can be written

$$f(x) = m(x) + \lambda \int_0^1 \frac{D(x, y; \lambda)}{D(\lambda)} m(y)dg(y),$$

where

$$D(\lambda) = 1 - \lambda \int_0^1 K(s, s)dg(s) + \cdots,$$

$$D(x, y; \lambda) = K(x, y) - \lambda \int_0^1 \left| \begin{array}{cc} K(x, y) & K(x, s) \\ K(s, y) & K(s, s) \end{array} \right| dg(s) + \cdots.$$

The proof follows along the same lines as in the ordinary case. We now state a corollary of Theorem 2 that represents most of the known results concerning solutions of equation (1).

**Corollary.* If $|G(x, y_2) - G(x, y_1)| \leq |g(y_2) - g(y_1)|$, then, excepting characteristic values, equation (1) has (3) as a solution.

Any result for the ordinary Fredholm equation carries a related result for equation (1). To see this, we assume without loss of generality that $g(y_1) < g(y_2)$ if $y_1 < y_2$, and apply to (2) the transformation $\beta(s) = \lim sup E_y(s \geq g(y))$,

$$\beta(s) = \lim sup E_y(s \geq g(y)), \quad g(0) \leq s \leq g(1),$$

$$f(x) = m(x) + \lambda \int_0^1 K(x, y)f(y)dg(y)$$

$$f(x) = m(x) + \lambda \int_{\beta(0)}^{\beta(1)} K(x, \beta(s))f(\beta(s))ds.$$
If we let $\omega$ be any of the possible solutions of
$$x = \beta(\omega), \quad g(0) \leq \omega \leq g(1),$$
we may write (4) in the form
$$F(\omega) = M(\omega) + \lambda \int_{\varphi(0)}^{\varphi(1)} k(\omega, s)F(s)ds,$$
where $F(\omega) = f(\beta(\omega))$, $M(\omega) = m(\beta(\omega))$, $k(\omega, s) = K(\beta(\omega), \beta(s))$. We thus have our main result:

**Theorem 3.** When $G(x, y)$ is absolutely continuous $g(y)$ the Fredholm-Stieltjes integral equation (1) is reducible to an ordinary Fredholm integral equation.

4. **Mixed linear equations.** The mixed equation*

$$f(x) = m(x) + \sum_{i=1}^{m} \lambda K^{(i)}(x)f(s_i) + \lambda \int_{0}^{1} K(x, s)f(s)ds$$

can easily be put into the form

$$f(x) = m(x) + \lambda \int_{0}^{1} R(x, s)f(s)ds.$$

Thus from Theorem 3 we see that equation (5) is reducible to a Fredholm integral equation.

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**A THEOREM ON QUADRATIC FORMS†**

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In this note the following result is proved:

**Theorem.** Suppose $A[x] = a_{\alpha\beta}x_{\alpha}x_{\beta}$, $B[x] = b_{\alpha\beta}x_{\alpha}x_{\beta}$ are real quadratic forms in $(x_{\alpha})$, $(\alpha = 1, \ldots, n)$, and that $A[x] > 0$ for all real $(x_{\alpha}) \neq (0_{\alpha})$ satisfying $B[x] = 0$. Then there exists a real constant $\lambda_0$ such that $A[x] - \lambda_0B[x]$ is a positive definite quadratic form.

This theorem is of use in considering the Clebsch condition for multiple integrals in the calculus of variations. A. A. Albert§ has given

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‡ The tensor analysis summation convention is used throughout.