NOTE ON AN ELEMENTARY PROBLEM OF INTERPOLATION

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The unique polynomial of degree \( n - 1 \) assuming the values \( y_1, y_2, \ldots, y_n \) at the abscissas \( x_1, x_2, \ldots, x_n \), respectively, is given by the Lagrange interpolation formula

\[
L_n(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots + y_n l_n(x),
\]

where

\[
l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \quad k = 1, 2, \ldots, n,
\]

(fundamental polynomials of the Lagrange interpolation) and the polynomial \( \omega(x) \) is defined by

\[
\omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n),
\]

where \( c \) denotes an arbitrary constant not equal to zero.

In this note we prove the following theorem:

**Theorem.** In the Lagrange interpolation formula let \( x_k = x_k^{(n)} = \cos \left( \frac{(2k-1)\pi}{2n} \right) = \cos \theta_k^{(n)}, \quad (k=1, 2, \ldots, n) \), which implies \( \omega(x) = T_n(x) = \cos (n \text{arc cos } x) = \cos n\theta \) (Tschebycheff polynomial). Then

\[
|l_k^{(n)}(x)| = \left| \frac{\omega(x)}{\omega'(x_k)(x - x_k)} \right| < \frac{4}{\pi}, \quad -1 \leq x \leq +1,
\]

for all \( n \) and \( k \), and furthermore

\[
\lim_{n \to \infty} \left| l_1^{(n)}(+1) \right| = \lim_{n \to \infty} \left| l_n^{(n)}(-1) \right| = \frac{4}{\pi}.
\]

In this connection Fejér* proved for all \( n, k, \) and \( x, ( -1 \leq x \leq +1 ) \),

\[
|l_k^{(n)}(x)| < 2^{n/2}.
\]

Of course (5) implies that inequality (4) is the best possible in the following sense: For any \( \epsilon > 0 \) there exist values of \( n, k, \) and \( x, \)

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* L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55; see pp. 10, 11. This paper will hereafter be referred to as L.
(-1 ≤ x ≤ +1), such that

\[ |l_k^{(n)}(x)| > \frac{4}{\pi} - \varepsilon. \tag{7} \]

Our proof depends upon the Hermite interpolation formula which gives the unique polynomial \( H(x) \) of degree \( 2n - 1 \) satisfying the conditions

\[ H(x_k) = y_k, \quad H'(x_k) = y'_k, \quad k = 1, 2, \ldots, n, \tag{8} \]

where the \( y_k \) and \( y'_k \) are given numbers. It is easy to show that*

\[ H(x) = \sum_{k=1}^{n} y_k v_k(x) \left\{ l_k(x) \right\}^2 + \sum_{k=1}^{n} y'_k (x - x_k) \left\{ l_k(x) \right\}^2, \tag{9} \]

where

\[ v_k(x) = 1 - (x - x_k) \frac{\omega''(x_k)}{\omega'(x_k)}, \tag{10} \]

\[ \sum_{k=1}^{n} v_k(x) \left\{ l_k(x) \right\}^2 = 1. \tag{11} \]

For the Tschebycheff abscissas we have

\[ v_k(x) = v_k^{(n)}(x) = \frac{1 - x x_k^{(n)}}{1 - (x x_k^{(n)})^2}, \quad x_k^{(n)} = \cos (2k - 1)\pi/2n. \tag{12} \]

Fejér proved (6) by aid of the simple inequality \( v_k^{(n)}(x) \geq 1/2. \)

We also need the following result due to M. Riesz:

**LEMMA.** A trigonometric polynomial of degree \( n - 1 \) assumes the maximum of its absolute value at a point whose distance from any of the roots of this trigonometric polynomial is not less than \( \pi/[2(n - 1)] \).

We are now in position to prove the theorem. For \( n = 1 \) and \( n = 2 \)

\[ \left| l_1^{(1)}(x) \right| = 1, \quad \left| l_1^{(3)}(x) \right| = \left| \frac{\sin \pi/4 \cos 2\theta}{2(\cos \theta - \cos \pi/4)} \right| = \frac{1}{2^{1/2}} \left| \cos \theta + \frac{1}{2^{1/2}} \right| < \frac{1 + 2^{1/2}}{2} < \frac{4}{\pi}, \]


† See L., p. 5.

Thus we have to consider only the case $n \geq 3$. For the Tschebycheff abscissas we have *

$$l_k^{(n)}(x) = (-1)^{k+1} n^{-1} \sin \theta_k^{(n)} \frac{\cos n\theta}{\cos \theta - \cos \theta_k^{(n)}}, \quad x = \cos \theta.$$  

From (13) it follows that $l_k^{(n)}(\cos \theta)$ is a trigonometric polynomial of degree $n - 1$. For $2 \leq k \leq n - 1$, the roots of $l_k^{(n)}(\cos \theta)$ in $(0, \pi)$ are

$$\theta_{\nu}^{(n)} = (2\nu - 1) \frac{\pi}{2n}, \quad 1 \leq \nu \leq n, \quad \nu \neq k,$$

and since $\theta_{\nu + 1}^{(n)} - \theta_{\nu}^{(n)} = \pi/n$, $\theta_k^{(n)} - \theta_0^{(n)} = \pi/2n$, $|l_k^{(n)}(\cos \theta)|$ assumes its maximum between $\theta_k^{(n)}$ and $\theta_{k+1}^{(n)}$. Further it is clear that $|l_1^{(n)}(\cos \theta)|$ and $|l_n^{(n)}(\cos \theta)|$ assume their maxima at $\theta = 0$ and $\theta = \pi$, respectively. Let us consider first $l_1^{(n)}(x)$ and $l_n^{(n)}(x)$. According to the last remark it will be sufficient to find bounds for $|l_1^{(n)}(+1)|$ and $|l_n^{(n)}(-1)|$. From (13) we have

$$|l_1^{(n)}(+1)| = |l_n^{(n)}(-1)| = \frac{\sin \theta_1^{(n)}}{n(1 - \cos \theta_1^{(n)})} = \frac{1}{n} \cot \frac{\pi}{4n},$$

whence

$$\lim_{n \to \infty} |l_1^{(n)}(+1)| = \lim_{n \to \infty} |l_n^{(n)}(-1)| = \frac{4}{\pi}.$$ 

By differentiation we easily see that $x \cot x$ decreases if $x$ increases so that

$$|l_1^{(n)}(+1)| \leq |l_1^{(n+1)}(+1)|, \quad |l_n^{(n)}(-1)| \leq |l_{n+1}^{(n+1)}(-1)|.$$ 

From (15) and (16) we obtain (7), that is, the second part of the statement.

We now prove that

$$\max_{-1 \leq x \leq +1} |l_k^{(n)}(x)| < |l_1^{(n)}(+1)|, \quad 2 \leq k \leq n - 1.$$ 

By (16) it suffices to show that

$$\max_{-1 \leq x \leq +1} |l_k^{(n)}(x)| = |l_k^{(n)}(\mu_k)| < |l_1^{(n)}(+1)| = \frac{1}{2}(1 + 2^{1/2}).$$

* See L, p. 5.
In order to prove (18) we show that

\[ v_k(n)(\mu_k) > \frac{13}{18}, \quad 2 \leq k \leq n - 1. \]

Then (11) furnishes, since \( v_k(x) \geq 0, \)

\[ 1 = \sum_{k=1}^{n} v_k(n)(\mu_k) \left\{ I_k(n)(\mu_k) \right\}^2 \geq \frac{13}{18} \left\{ I_k(n)(\mu_k) \right\}^2, \]

that is,

\[ \left| I_k(n)(\mu_k) \right| < \left[ \frac{18}{13} \right]^{1/2} < (1.4)^{1/2} < 1.2 = \frac{1}{2}(1 + 1.4) < \frac{1}{2}(1 + 2^{1/2}). \]

Let \( \mu_k = \cos \phi, \) \( 0 < \phi < \pi. \) According to the lemma we have \( |\phi - \theta_k^{(n)}| < \pi/2n. \) On account of (12) it is sufficient to prove

\[ 1 - \cos \theta_k^{(n)} \cos (\theta_k^{(n)} + \delta) \geq \frac{13}{18}, \quad \delta = \pm \frac{\pi}{2n}. \]

We can assume that \( \theta_k^{(n)} < \pi/2 \) and \( \delta = -\pi/2n. \) If we write \( \theta_k^{(n)} = 3\mu, \) we have \( \theta_k^{(n)} + \delta \geq 2\mu \) and \( \cos \mu = t, \) \( \frac{1}{2} \leq t \leq +1, \) hence

\[ \frac{\cos (\theta_k^{(n)} + \delta) - \cos \theta_k^{(n)}}{1 - \cos \theta_k^{(n)}} \leq \frac{\cos 2\mu - \cos 3\mu}{1 - \cos 3\mu} = \frac{4t^2 + 2t - 1}{4t^2 + 4t + 1}. \]

The last fraction is \( \leq 5/9 \) so that

\[ \frac{\cos \theta_k^{(n)}}{1 + \cos \theta_k^{(n)}} \frac{\cos (\theta_k^{(n)} + \delta) - \cos \theta_k^{(n)}}{1 - \cos \theta_k^{(n)}} \leq \frac{5}{18}, \]

which is equivalent to (20). This completes the proof.

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