REMARKS ON THE CLASSICAL INVERSION FORMULA FOR THE LAPLACE INTEGRAL

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If a function \( f(s) = f(s + i\tau) \) is defined for \( \sigma > 0 \) by the Laplace integral

\[
f(s) = \int_0^\infty e^{-st}\phi(t)\,dt,
\]

then the classical inversion formula is

\[
\phi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s) e^{st}\,ds, \quad c > 0, \ t > 0.
\]

Conditions for the validity of this formula have frequently been discussed. However, the authors know of no adequate treatment* of the case when \( \phi(t) \) belongs to \( L^2 \) in \((0, \infty)\):

\[
\int_0^\infty |\phi(t)|^2\,dt < \infty.
\]

We employ here the usual notation,

\[
l.i.m. \phi_{a}(t) = \phi(t), \quad a \to \infty
\]

to mean that \( \phi_{a}(t) \) and \( \phi(t) \) belong to \( L^2 \) in \((-\infty, \infty)\) and that

\[
\lim_{a \to \infty} \int_{-\infty}^{\infty} |\phi_{a}(t) - \phi(t)|^2\,dt = 0.
\]

It is clear first that if (3) holds then (1) converges absolutely for \( \sigma > 0 \), since

\[
|f(\sigma + i\tau)|^2 = \left| \int_0^\infty e^{-st}\phi(t)\,dt \right|^2 \leq \int_0^\infty e^{-2\sigma t}\,dt \int_0^\infty |\phi(t)|^2\,dt.
\]

Moreover, by the Plancherel theorem regarding Fourier transforms,

\[
l.i.m. \int_{\sigma}^{\infty} e^{-ir\phi(t)}\,dt
\]

exists. We denote it by \( f(\tau) \). The same theorem gives us at once that

\[
\lim_{a \to \infty} \frac{1}{2\pi} \int_{-a}^{a} f(\tau) e^{i\tau t} d\tau = \begin{cases} 
\phi(t), & t > 0, \\
0, & t < 0,
\end{cases}
\]

or

\[
\lim_{a \to \infty} \frac{1}{2\pi i} \int_{-ia}^{ia} f(s) e^{s \tau} ds = \begin{cases} 
\phi(t), & t > 0, \\
0, & t < 0.
\end{cases}
\]

Hence (2) with \( c = 0 \) is valid in the sense of (4). However, if \( c > 0 \), (2) is no longer valid in this sense.

If \( c > 0 \), it is again clear from the Plancherel theorem that

\[
\lim_{a \to \infty} \frac{1}{2\pi} \int_{-a}^{a} f(c + \tau) e^{i\tau t} d\tau = \begin{cases} 
e^{-ct}\phi(t), & t > 0, \\
0, & t < 0.
\end{cases}
\]

But this does not imply that

\[
\lim_{a \to \infty} \frac{1}{2\pi i} \int_{-ia}^{ia} f(s) e^{s \tau} ds = \begin{cases} 
\phi(t), & t > 0, \\
0, & t < 0,
\end{cases}
\]

unless \( f(s) \) is identically zero. For, set \( \phi_a(t) = \int_{-a}^{a} f(c + \tau) e^{i\tau t} d\tau \). It will be sufficient to show that

\[
\int_{-\infty}^{\infty} e^{2ct} | \phi_a(t) |^2 dt = \infty
\]

for some \( a \). Choose \( a \) so that

\[
f(c + ia) \neq f(c - ia).
\]

This is possible, for otherwise we should have by use of (1) that

\[
\int_{0}^{\infty} e^{-ct}\phi(t) \sin at dt = 0
\]

for all \( a \). By the uniqueness theorem for the Fourier sine transform this would imply that \( \phi(t) \) is equivalent to zero and that \( f(s) \) is identically zero. In fact we see that (6) may be satisfied for some \( a \) in every interval however small.

An integration by parts of the integral defining \( \phi_a(t) \) gives

\[
\phi_a(t) = \frac{f(c + ia) e^{iat} - f(c - ia) e^{-iat}}{it} - \frac{1}{t} \int_{-a}^{a} e^{i\tau t} f'(c + \tau) d\tau
\]

\[
= \frac{f(c + ia) e^{iat} - f(c - ia) e^{-iat}}{it} + o \left( \frac{1}{|t|} \right), \quad |t| \to \infty,
\]
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\[ \frac{e^{iat}}{it} \left[ \{f(c + ia) - f(c - ia)\} + f(c - ia)(1 - e^{-2iat}) \right] \]

\[ + o \left( \frac{1}{|t|} \right). \]

Hence

\[ t^2 | \phi_a(t) |^2 \geq \left[ \frac{k}{2} - 2l \left| \sin at \right| \right]^2, \quad t > t_0, \]

where \( t_0 \) is a sufficiently large positive number and

\[ k = |f(c + ia) - f(c - ia)|, \quad l = |f(c - ia)|. \]

Since \( 2l|\sin at| < k/4 \) in an interval of length \( \delta \), say, about \( t = 0 \) and in intervals congruent to this one, modulo \( \pi/\alpha \), it is clear that the integrand of (5) exceeds \( k^2e^{2\alpha t}/16\alpha^2 \) in infinitely many intervals of length \( \delta \). This is sufficient to insure the divergence of the integral.

We collect our results in the following form:

**Theorem.** If \( \phi(t) \) belongs to \( L^2 \) in \( (0, \infty) \), then it has a Laplace transform \( f(\sigma + it) \) defined for \( \sigma > 0 \) by the absolutely convergent integral

\[ f(\sigma + it) = \int_0^\infty e^{-(\sigma + it)t} \phi(t) dt, \]

and for \( \sigma = 0 \) by

\[ f(it) = \lim_{\sigma \to 0} \int_0^\infty e^{-\sigma t} \phi(t) dt. \]

The inversion formula

\[ 1.\text{i.m.} \left. \frac{1}{2\pi i} \int_{c - ia}^{c + ia} f(s) e^{st} ds = \right\{ \begin{array}{ll} \phi(t), & t > 0, \\ 0, & t < 0, \end{array} \]

is false \( (f(s) \neq 0) \) for \( c > 0 \) and valid for \( c = 0 \). For all \( c \geq 0 \)

\[ 1.\text{i.m.} \left. \frac{1}{2\pi} \int_{-\alpha}^{\alpha} f(c + it) e^{i\tau \phi} d\tau = \right\{ \begin{array}{ll} e^{-\alpha t} \phi(t), & t > 0, \\ 0, & t < 0. \end{array} \]

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