POLYGENIC FUNCTIONS WHOSE ASSOCIATED 
ELEMENT-TO-POINT TRANSFORMATION 
CONVERTS UNIONS INTO POINTS*

EDWARD KASNER

1. Introduction. A function \( w = \phi(x, y) + i\psi(x, y) \) is called a polygenic function of the complex variable \( z = x + iy \) if the real functions \( \phi \) and \( \psi \) are general, that is, are not required to satisfy the Cauchy-Riemann equations. The value of the derivative of a polygenic function at a point \( z_0 \) depends in general not only on the point \( z_0 \) but also on the direction \( \theta \) along which \( z \) approaches \( z_0 \); that is, \( dw/dz \) is of the form \( F(x, y, \theta) \). Thus the derivative \( \gamma = dw/dz \) of a polygenic function may be regarded as determining a correspondence between the lineal elements \( (x, y, \theta) \) of the \( z \)-plane and the points \( (\alpha, \beta) \) of the \( \gamma \)-plane, where \( \gamma = \alpha + i\beta \). We call this correspondence the element-to-point transformation \( T \) associated with the polygenic function \( w \).

In previous papers (Kasner, A new theory of polygenic functions, Science, vol. 66 (1927), pp. 581–582; General theory of polygenic functions, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82; The second derivative of a polygenic function, Transactions of this Society, vol. 30 (1928), pp. 805–818) we have shown that the element-to-point transformation \( T \) associated with a polygenic function must possess the two following properties:

I. Elements at a given point in the \( z \)-plane correspond to points of a circle \( I \) in the \( \gamma \)-plane.

II. Corresponding central angles of the circle and angles at the point are in the ratio \( -2:1 \).

If an element-to-point transformation \( T \) possesses the property I, then we define the function \( H + iK \), which as a vector represents the center of the circle \( I \), to be the center function of \( T \), and the function \( (H + k) + i(K + k) \), which as a vector represents the point (called the initial point of the circle) on the circle \( I \) which corresponds to the initial direction \( \theta = 0 \) in the \( z \)-plane, we define to be the principal phase function of \( T \). The circle \( I \) together with its initial point we call a clock.

We then find (Kasner, A complete characterization of polygenic functions, Proceedings of the National Academy of Sciences, vol. 22

* Presented to the Society, September 6, 1938.
(1936), pp. 172–177), that the element-to-point transformation \( T \) associated with a polygenic function possesses the following additional property:

**III. The principal phase point of the clock representing the derivative of the center function of \( T \) coincides with the center of the clock representing the derivative of the principal phase function of \( T \).**

In the paper last cited, it is also proved that for an element-to-point transformation to be associated with a polygenic function, it is necessary and sufficient that it possess the properties I, II, and III.

The associated transformation \( T \) carries a single element into a point, and it carries the \( \infty \) elements at a point in the \( \mathbb{C} \)-plane into the points of a circle in the \( \mathbb{C} \)-plane (property I). However, a given point in the \( \mathbb{C} \)-plane will correspond, in general, not to a single element in the \( \mathbb{C} \)-plane, but to a series (\( \infty \) elements). Now we inquire under what conditions will this series be a union (curve or point). Of course we mean that this shall happen for all the points of the \( \mathbb{C} \)-plane, that is, we demand that all the series so formed shall be unions. It turns out analytically that this problem means that a certain pair of functions of \((x, y, p)\) shall be in involution. In our discussion, we do not demand that the jacobians be different from zero; therefore our solution will include degenerate cases. But actually the major part of the solution is not degenerate.

Our problem is thus to determine a certain specific class of polygenic functions, namely, that class for which, instead of associated series, we obtain unions. This class, we find by a long analytic discussion, consists of the following three distinct types:

(A) The monogenic functions \( w = f(z) \).

(B) The mixed quadratic fractional polygenic functions

\[
w = -\frac{az + b}{\alpha(\alpha z + b)} + cz + d, \quad \alpha \neq 0.
\]

(C) The affine linear polygenic functions \( w = Az + Bz + C, (B \neq 0) \).

Of these three types, the quadratic type (B) is the essentially significant result revealed by our investigation.

2. The associated element-to-point transformation \( T \) of a polygenic function. Let the element-to-point transformation \( T \)

\[
\gamma = \alpha(x, y, \theta) + i\beta(x, y, \theta)
\]

possess the properties I, II, and III. Then we find that \( T \) can be written in the form
\[ \alpha = H + h \cos 2\theta + k \sin 2\theta, \]
\[ \beta = K - h \sin 2\theta + k \cos 2\theta, \]
where \( H, K, h, k \) are functions of \( x \) and \( y \) only which satisfy
\[ H_x - K_y = h_x + k_y, \quad K_x + H_y = k_x - h_y. \]

Let \( w = \phi(x, y) + i\psi(x, y) \) be any polygenic function to which \( T \) is the associated element-to-point transformation. Then \( w \) must satisfy the two equations
\[ \frac{1}{2} \left[ \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right] w = H + iK, \quad \frac{1}{2} \left[ \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right] w = h + ik. \]

From (4) it can very easily be proved that any two polygenic functions which have the same associated element-to-point transformation \( T \) differ merely by a complex constant.

3. Polygenic functions whose associated element-to-point transformations convert unions into points. We prove the following theorem:

**Theorem.** The totality of polygenic functions whose associated element-to-point transformations convert unions of the \( z \)-plane into the points of the \( \gamma \)-plane consists of the three distinct types \( (A), (B), (C) \), as specified at the end of §1.

The proof will occupy the next three pages. Upon writing \( p = \tan \theta \) the equations (2) become
\[ \alpha = \frac{(H + h) + 2kp + (H - h)p^2}{1 + p^2}, \]
\[ \beta = \frac{(K + k) - 2hp + (K - k)p^2}{1 + p^2}. \]

First let us consider the case in which \( h \) and \( k \) are both zero. Then from (5) we see that our element-to-point transformation becomes a point-to-point transformation. Hence when \( h = k = 0 \), the points of the \( z \)-plane become the points of the \( \gamma \)-plane, and the condition of our theorem is therefore satisfied. From (4) we find that \( w \) must be a monogenic function of \( z \). Henceforth we shall suppose that at least one of the functions \( h \) and \( k \) is different from zero.

For the element-to-point transformation (5) to convert unions of the \( z \)-plane into the points of the \( \gamma \)-plane it is necessary and sufficient that
that is, the functions $\alpha$ and $\beta$ must be in involution. Substituting (5) into (6) and making use of the equations (3), we obtain

\[
\frac{(K_x + k_x) + p(H_x - 3h_x) + p^2(-H_y - 3k_y) + p^3(K_y - k_y)}{(H_x + h_x) + p(-K_x + 3k_x) + p^2(K_y + 3k_y) + p^3(H_y - h_y)} = \frac{h + 2kp - hp^2}{-k + 2hp + kp^2}.
\]

Since the equation (7) is an identity in $p$, we obtain, upon setting the coefficients $c$ of the powers of $p$ equal to zero and making use of the equations (3), the equations

\[
\begin{align*}
h = & -K_x - k_x \\
\frac{h}{k} = & \frac{H_x + k_x}{H_x + h_x} = \frac{3H_x - h_x}{3K_x - k_x} = \frac{H_y + k_y}{K_y - h_x} = \frac{3H_y + h_y}{3K_y + k_y} = \frac{K_y - h_x}{-H_y + h_y}.
\end{align*}
\]

From the equations (8) it follows by ratio and proportion that

\[
\frac{h}{k} = \frac{H_y - K_x}{H_x + K_y} = \frac{H_x + K_y}{-H_y + K_x}.
\]

Since all the functions are real, it follows from (9) that

\[
H_x = -K_y, \quad H_y = K_x.
\]

From (10) we find that $H + iK$ is an analytic function of $\bar{z}$; that is

\[
H + iK = \lambda(\bar{z}),
\]

where $\lambda(\bar{z})$ is an analytic function of $\bar{z}$.

Substituting $K_x = H_y$ and $K_y = -H_x$ into (8), we find that the equations (8) become

\[
\begin{align*}
\frac{h}{k} = & -H_y - k_x \\
\frac{h}{k} = & \frac{H_x + k_x}{H_x + h_x} = \frac{3H_x - h_x}{3H_y - k_x} = \frac{H_x + k_y}{H_y - h_y} = \frac{3H_y + h_y}{3H_x + k_y}.
\end{align*}
\]

Also substituting $K_x = H_y$ and $K_y = -H_x$ into equations (3), we obtain

\[
2H_x = h_x + k_y, \quad 2H_y = -h_y + k_x.
\]
Upon substituting (13) into (12), we find
\[
\frac{h}{k} = \frac{h_y - 3k_z}{3h_x + k_y} = \frac{h_x + 3k_y}{-3h_y + k_x}.
\]
These two equations are equivalent to the equations
\[
3hh_x + 3kk_x = kh_y - kh_y, \\
3hh_y + 3kk_y = kh_x - kh_x.
\]
The equations (15) are then equivalent to the equations
\[
\frac{3}{2} \frac{\partial}{\partial x} \log (h^2 + k^2) = -\frac{\partial}{\partial y} \arctan k/h,
\]
\[
\frac{3}{2} \frac{\partial}{\partial y} \log (h^2 + k^2) = \frac{\partial}{\partial x} \arctan k/h.
\]
From (16), it follows that \((3/2) \log (h^2+k^2) + i \arctan k/h\) is an analytic function of \(\tilde{z}\). Thence \(\exp \{ (3/2) \log (h^2+k^2) + i \arctan k/h \}\) is an analytic function of \(\tilde{z}\); that is
\[
(h^2 + k^2)(h + ik) = \mu(\tilde{z}),
\]
where \(\mu(\tilde{z})\) is an analytic function of \(\tilde{z}\). Moreover \(\mu(\tilde{z}) \neq 0\), since at least one of the quantities \(h, k\) is different from zero.
Now (17) may be written in the form
\[
(h - ik)(h + ik)^2 = \mu(\tilde{z}).
\]
Upon taking the conjugate of the equation (18), we obtain
\[
(h + ik)(h - ik)^2 = \mu(z).
\]
Solving the equations (18) and (19) for \(h + ik\), we find
\[
h + ik = [\mu(\tilde{z})]^{2/3}/[\mu(z)]^{1/3}.
\]
It is seen that the condition of our theorem is satisfied if the four functions \(H, K, h, k\) satisfy the equations (3), (11), and (20). From these equations, we find that \(w\) must be an analytic polygenic function of \(x\) and \(y\). Hence \(w\) may be written as an analytic function of \(z\) and \(\tilde{z}\); that is
\[
w = f(z, \tilde{z}),
\]
where \(f\) is an analytic function of \(z\) and \(\tilde{z}\).
From equations (4), (11), (20), and (21), we find that
\[
f_x = \lambda(z), \quad f_\tilde{z} = [\mu(\tilde{z})]^{2/3}/[\mu(z)]^{1/3}.
\]
The equations (3) are then equivalent to

\[ \frac{\lambda'(z)}{[\mu(z)]^{2/3}} = \frac{d}{dz} \left[ \mu(z) \right]^{-1/3}. \]

From (23) we find that

\[ \frac{\lambda'(z)}{[\mu(z)]^{2/3}} = a, \quad \frac{d}{dz} \left[ \mu(z) \right]^{-1/3} = a, \]

where \( a \) is a complex constant. From (24) we obtain

\[ \lambda'(z) = \frac{a}{(\bar{a}z + \bar{b})^2}, \quad \mu(z) = \frac{1}{(\bar{a}z + \bar{b})^3}. \]

First let us suppose that \( a \) is zero. Then from (22) and (25) we find that \( f_z \) and \( f_\gamma \) are both constants. Thus \( w \) is the affine linear polygenic function \( w = Az + Bz + C \) where \( B \neq 0 \).

Next let \( a \neq 0 \). Then from (22) and (25) we find that

\[ f_z = -\frac{a}{a(\bar{a}z + \bar{b})} + c, \quad f_\gamma = \frac{az + b}{(\bar{a}z + \bar{b})^2}. \]

From (26) we see that our polygenic function \( w \) must be the mixed quadratic fractional polygenic function

\[ w = -\frac{az + b}{a(\bar{a}z + \bar{b})} + cz + d, \]

where \( a \neq 0, b, c, d \) are complex constants. This completes the proof.

4. The unions which under the associated element-to-point transformation become points. We consider the three classes of functions mentioned in the theorem.

(A) The monogenic functions \( w = f(z) \). Let the monogenic function \( w = f(z) \) be not an affine linear monogenic function. Then the elements at any point \( z \) of the \( z \)-plane are converted into a point \( \gamma \) of the \( \gamma \)-plane and conversely. Thus, for a monogenic function which is not affine linear, the \( \infty^2 \) point-unions of the \( z \)-plane are converted into the \( \infty^2 \) points of the \( \gamma \)-plane, and conversely.

On the other hand, let \( w \) be an affine linear monogenic function. Then the derivative of \( w \) is constant; hence in the \( \gamma \)-plane we have a single fixed point. To this fixed point corresponds the opulence (the totality of \( \infty^2 \) elements) of the \( z \)-plane. Thus for an affine linear monogenic function, the opulence of the \( z \)-plane is converted into a fixed point of the \( \gamma \)-plane.
In the geometry of lineal elements of the plane, a set of $\infty^1$ elements is called a series, a set of $\infty^2$ elements is called a field, and the totality of $\infty^3$ elements is called the opulence.

(B) The mixed quadratic fractional polygenic functions

$$w = - \frac{az + b}{\bar{a}(\bar{a}z + \bar{b})} + cz + d.$$  

The unions in the $s$-plane, which under the associated element-to-point transformation of the polygenic function $w$ become the points of the $\gamma$-plane, are the $\infty^2$ circles through the point $-b/a$ and the field defined by the $\infty^1$ straight lines through the point $-b/a$.

To a point $\gamma \neq c$ of the $\gamma$-plane there corresponds a definite circle of the $s$-plane through the point $-b/a$, and conversely. The center $C$ and the radius $R$ are given by the formulas

$$C = -\frac{b}{a} + \frac{\bar{a}}{a^2(\bar{c} - \bar{\gamma})}, \quad R^2 = \frac{1}{a\bar{a}(c - \gamma)(\bar{c} - \bar{\gamma})}.$$  

The field defined by the pencil of straight lines through the point $-b/a$ of the $s$-plane is converted into the point $c$ of the $\gamma$-plane.

(C). The affine linear polygenic functions $w = Az + B\bar{z} + C$, $(B \neq 0)$. The associated element-to-point transformation of the affine linear polygenic function $w = Az + B\bar{z} + C$, $(B \neq 0)$, converts the opulence (the totality of $\infty^3$ elements) of the $s$-plane into the $\infty^1$ points of the circle in the $\gamma$-plane whose center is $A$ and whose radius is $|B| \neq 0$.

It is found that to any point of the fixed circle in the $\gamma$-plane, there corresponds the field defined by $\infty^1$ parallel straight lines, and conversely.

5. Scholium. We thus find that there are four distinct geometric possibilities in the $s$-plane:

(A'). The $\infty^2$ point-unions (stars).

(A''). The opulence of elements in the $s$-plane.

(B). The $\infty^2$ circles through a fixed point together with the field defined by the pencil of straight lines through the same fixed point.

(C). The $\infty^1$ fields defined by parallel straight lines.

In the $\gamma$-plane, we find the following three distinct geometric possibilities:

(A', B). The $\infty^2$ points.

(C). The $\infty^1$ points of a fixed circle.

(A'') A single fixed point.

We remark in conclusion that the quadratic type (B), formula (27), gives the really significant configuration.