ON THE ABSOLUTE SUMMABILITY OF FOURIER SERIES*

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1. Introduction. A series \( \sum u_n \) is said to be absolutely summable by a method \( a \) defined by a matrix \( a_{mn} \) if

\[
\sum_{m=1}^{\infty} \left| S_m(a, u) - S_{m-1}(a, u) \right| < \infty,
\]

where

\[
S_m(a, u) = \sum_{n=0}^{\infty} a_{mn}u_n.
\]

Similarly a series is said to be absolutely summable \( |A| \) if

\[
S(r, u) = \sum_{n=0}^{\infty} u_nr^n \in BV \quad \text{on} \quad (0, 1).
\]

It is known that if \( \sum u_n \) is absolutely summable \( |C_\alpha| \) for some \( \alpha > 0 \), then it is absolutely summable \( |A| \). There are, however, series absolutely summable \( |A| \) but not \( |C_\alpha| \) for any \( \alpha \) whatever. We intend to give here an example of a Fourier series with that property.

Bosanquet† has proved that, if the Fourier series of \( f(x) \) is absolutely summable \( |C_\alpha| \), then the function

\[
\phi_\beta(f, t) = \beta t^{-\beta} \int_0^t \{f(x + \tau) + f(x - \tau) - 2f(x)\} (t-\tau)^{\beta-1}d\tau
\]

is of bounded variation on \((0, \pi)\) for \( \beta > \alpha \); and conversely, if \( \phi_\alpha(t) \) is of bounded variation, the Fourier series of \( f(x) \) is absolutely summable \( |C_\beta| \), \( (\beta > \alpha + 1) \).

2. Preliminary definitions. Let \( \alpha_{nk}, \beta_{nk} \) be defined for \( n = 1, 2, \ldots, k = 1, 2, \ldots, n \), by

\[
\alpha_{nk} = 2^{-k-n-n/(k-1/2)}, \quad \beta_{nk} = 2^{-n} - 2^{-n-n/(k-1/2)}.
\]

Then, since \( k \leq n \), we have

\[
\beta_{nk} > 2^{-n-1}.
\]

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Let \( f_{nk}(x) \) be defined over \(-\pi \leq x \leq \pi\), so that

\begin{align*}
(2) \quad f_{nk}(x) &= 2^n, \quad \beta_{nk} \leq |x| \leq \beta_{nk} + \alpha_{nk}, \\
(3) \quad f_{nk}(x) &= -f_{nk}(x - 2i\alpha_{nk}), \quad \beta_{nk} + 2i\alpha_{nk} < |x| \leq \beta_{nk} + 2^{i+1}\alpha_{nk}, \\
(4) \quad f_{nk}(x) &= 0 \quad \text{elsewhere on } (-\pi, \pi),
\end{align*}

where in (3) \( j \) takes on the values 0, \ldots, \( k - 1 \). The relation (3) implies that

\[
\int_{\beta_{nk}}^{\beta_{nk} + 2\alpha_{nk}} f_{nk}(x)\,dx = \int_{\beta_{nk}}^{\beta_{nk} + \alpha_{nk}} f_{nk}(x)\,dx - \int_{\beta_{nk}}^{\beta_{nk} + \alpha_{nk}} f_{nk}(x)\,dx = 0,
\]

and by induction

\[
\int_{\beta_{nk}}^{\beta_{nk} + 2j\alpha_{nk}} f_{nk}(x)\,dx = 0, \quad 1 \leq j \leq k.
\]

If we define

\[
\Phi_1(f, t) = \int_0^t \{f(x) + f(-x) - 2f(0)\}\,dx,
\]

we can obtain the following relations analogous to (3) and (4):

\begin{align*}
(5) \quad \Phi_1(f_{nk}, t) &= -\Phi_1(f_{nk}, t - 2^{i+1}\alpha_{nk}), \quad \beta_{nk} + 2^{i+1}\alpha_{nk} < x \leq \beta_{nk} + 2^{i+2}\alpha_{nk}, \\
(6) \quad \Phi_1(f_{nk}, t) &= 0 \quad \text{elsewhere on } (0, \pi),
\end{align*}

where in (5) \( j \) takes on the values 0, \ldots, \( k - 1 \).

We define by induction the functions

\[
\Phi_{r+1}(f, t) = (r + 1) \int_0^t \Phi_r(f, x)\,dx,
\]

for which it can be shown by similar reasoning that, for \( r \leq k \),

\begin{align*}
(7) \quad \Phi_r(f_{nk}, t) &= -\Phi_r(f_{nk}, t - 2^{r+1}\alpha_{nk}), \quad \beta_{nk} + 2^{r+1}\alpha_{nk} < x \leq \beta_{nk} + 2^{r+2}\alpha_{nk}, \\
(8) \quad \Phi_r(f_{nk}, t) &= 0, \quad \text{elsewhere on } (0, \pi),
\end{align*}

where in (7) \( j \) takes on the values 0, \ldots, \( k - r - 1 \). We notice that, at \( x = 0 \), \( \Phi_r(f, t) = t^{-r}\Phi_r(f, t) \), and therefore for \( r \leq k - 1 \)

\[
\Phi_r(f_{nk}, \beta_{nk} + \alpha_{nk}) = 2r(\beta_{nk} + \alpha_{nk})^{-r} \int_{\beta_{nk}}^{\beta_{nk} + \alpha_{nk}} 2^n(\beta_{nk} + \alpha_{nk} - x)^{-r-1}dx
\]

\[
= 2^{n+1}r(\beta_{nk} + \alpha_{nk})^{-r} \int_0^{\alpha_{nk}} (\alpha_{nk} - x)^{-r-1}dx
\]

\[
> 2^{n+1}r 2^{n/2} 2^{n/2} > 2^{-kr} 2^{n/2},
\]
since
\[
n(r + 1) - r\{n + n/(k - 1/2)\} = n\{1 - r/(k - 1/2)\} > n/2k.
\]
This shows that, for \( r < k \),
\[
T. V. (0, \pi) \phi_r(f_{nk}, x) > 2^{-kr} 2^{n/2k}.
\]
On the other hand,
\[
\phi'_k(f_{nk}, t) = kt^{-k} \Phi_{h-1}(f_{nk}, t) - kt^{-k-1} \Phi_k(f_{nk}, t),
\]
so that, if \( I = (\beta_{nk}, \beta_{nk} + 2^k \alpha_{nk}) = (\beta_{nk}, 2^{-n}) \), then
\[
\int_I \left| \phi'_k(f_{nk}, t) \right| dt = O \left\{ \int_0^{2^k \alpha_{nk}} 2^n (2^k \alpha_{nk} - t)^{-k} dt \right\}
\]
\[
+ 2^{(k+1)n} \int_0^{2^k \alpha_{nk}} 2^n (2^k \alpha_{nk} - t)^k dt \right\}
\]
\[
= O(2^{-n/2k}).
\]
Therefore
\[
T. V. (0, \pi) \phi_k(f_{nk}, t) = O(2^{-n/2k}).
\]

3. Definition of \( f(x) \). We define the functions
\[
f_k(x) = \sum_{[\log_2 k] + 1}^{\infty} f_{2^n + k, k}(x).
\]
For \( r \leq k \)
\[
\phi_r(f_{2^n + k, k}, t) \cdot \phi_r(f_{2^n + k, k}, t) = 0, \quad m \neq n,
\]
and therefore
\[
T. V. (0, \pi) \phi_r(f_k, t) = \sum_{[\log_2 k] + 1}^{\infty} T. V. (0, \pi) \phi_r(f_{2^n + k, k}, t) = \infty, \quad r < k,
\]
and
\[
(9) \quad T. V. (0, \pi) \phi_k(f_k, t) = \sum_{[\log_2 k] + 1}^{\infty} T. V. (0, \pi) \phi_k(f_{2^n + k, k}, t)
\]
\[
= O \left( \sum_{1}^{\infty} 2^{-j/2k} \right) = O(1).
\]
It follows then that, for \( s > k \),
\[
\phi_s(f_k, t) \in BV \quad \text{on} \quad (0, \pi).
\]
The Fourier series of \( f(x) \) must be absolutely summable \(|A|\), at \( x = 0 \). We set

\[
A_k = T.V. (0,1) \int_0^\pi f_k(t) \frac{1 - r^2}{1 - 2r \cos t + r^2} \, dt.
\]

A sequence \( d_k \) is then defined so that

\[
d_k \leq A_k 2^{-k},
\]

\[
d_k \leq 2^{-k} \int_0^\pi |f_k(x)| \, dx.
\]

The function

\[
f(x) = \sum_{k=1}^\infty d_k f_k(x)
\]

is the one we set out to construct. By (11), \( f(x) \in L \), since

\[
\int_0^\pi |f(x)| \, dx \leq \sum_{k=1}^\infty d_k \int_0^\pi |f_k(x)| \, dx \leq \sum_{k=1}^\infty 2^{-k} = 1.
\]

We have, by (10),

\[
T.V. (0,1) \frac{1}{\pi} \int_0^\pi f(t) \frac{1 - r^2}{1 - 2r \cos t + r^2} \, dt \leq \sum_{k=1}^\infty d_k A_k \leq \sum_{k=1}^\infty 2^{-k} = 1,
\]

which means that the Fourier series of \( f(x) \) is absolutely summable \(|A|\), at \( x = 0 \). Finally, using (9) we see that

\[
T.V. (0,\pi) \phi_j(f, t) > T.V. (0,\pi) \phi_j(f_j, t) - \left| \sum_{i=1}^{i-1} T.V. (0,\pi) \phi_j(f_k, t) \right|
\]

\[
= \infty - O(1) = \infty;
\]

so the Fourier series of \( f(x) \) cannot be \(|C_j|\) summable at \( x = 0 \), for any \( j \). This completes the proof of our assertion.

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