A TEST-RATIO TEST FOR CONTINUED FRACTIONS*

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Introduction. The general question of convergence of continued fractions of the form \( 1 + K_\infty \left[ b_n / 1 \right] \) remains in a large measure unanswered, even though continued fractions of this type are of especial importance from a function-theoretic point of view. Valuable contributions have been made by E. B. Van Vleck, A. Pringsheim, O. Szász, O. Perron, and others. Leighton and Wall [7] recently gave new types of convergence criteria for continued fractions of this kind. Jordan and Leighton in a paper to be published soon give a large number of new sets of sufficient conditions for convergence.

The purpose of the present paper is to establish the first test-ratio test for continued fractions and a very general theorem on convergence, which is also believed to be the first of its kind. This test leads to a class of continued fractions, the precise region of convergence of which is the interior of a circle. This is a new phenomenon.

1. A test-ratio test. Let

\[
1 + K_\infty \left[ b_n / 1 \right] = 1 + \frac{b_1}{1 + \frac{b_2}{1 + \cdots}}
\]

be a continued fraction in which the \( b_n \) are complex numbers \( \neq 0 \).

**Theorem 1.** If the ratio \( |b_{n+1}/b_n| \) is less than or equal to \( k < 1 \) for \( n \) sufficiently large, the continued fraction (1.1) converges at least in the wider sense. If \( |b_{n+1}/b_n| \) is greater than or equal to \( 1/k > 1 \) for \( n \) sufficiently large, the continued fraction diverges by oscillation. If the limit of the ratio is unity, the continued fraction may converge or diverge.

Suppose \( |b_{n+1}/b_n| \leq k < 1 \) for \( n \) sufficiently large. It follows that there exists a positive integer \( N \) such that \( |b_n| < 1/4 \) for \( n \geq N \). Each continued fraction \( K_\infty \left[ b_n / 1 \right] \) then converges (Van Vleck [2], Pringsheim [4]) for \( n \geq N \). The proof of the first statement of the theorem is complete.

Assume \( |b_{n+1}/b_n| \geq 1/k > 1 \) for \( n \) sufficiently large. Write (1.1) in the equivalent form (Perron [8], p. 197)

\[
1 + K_\infty \left[ 1/a_n \right],
\]

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where
\[ a_{2n} = \frac{b_1 b_3 \cdots b_{2n-1}}{b_2 b_4 \cdots b_{2n}}, \quad a_{2n-1} = \frac{b_2 b_4 \cdots b_{2n-2}}{b_1 b_3 \cdots b_{2n-1}}, \quad n = 1, 2, 3, \ldots; b_0 = 1. \]

It will be shown that the series \( \sum |a_n| \) converges, and it will follow from a theorem of Stern [1] that the continued fraction (1.2), and hence (1.1), diverges by oscillation. It is sufficient to observe that
\[
\left| \frac{a_{2n}}{a_{2n-2}} \right| = \left| \frac{b_{2n-1}/b_{2n}}{b_{2n-3}/b_{2n-2}} \right| \leq k < 1,
\]
\[
\left| \frac{a_{2n+1}}{a_{2n-1}} \right| = \left| \frac{b_{2n}/b_{2n+1}}{b_{2n-1}/b_{2n}} \right| \leq k < 1,
\]
for \( n \) sufficiently large. Thus the two series \( \sum |a_{2n+1}| \) and \( \sum |a_{2n}| \) converge. It follows that the series \( \sum |a_n| \) converges, and the second statement of the theorem follows as indicated.

To prove the final statement of the theorem, it is sufficient to consider the example
\[
(1.3) \quad 1 + \frac{a}{1 + \frac{a}{1 + \cdots}}.
\]
When \( a = 1 \) it is well known that this continued fraction converges to the value \((1 + 5^{1/2})/2\). When \( a = -1 \), a computation of the successive approximants proves immediately that the continued fraction diverges. Indeed, Szász [6] has shown that the continued fraction (1.3) diverges for every \( \epsilon > 0 \), if \( a = -\epsilon - 1/4 \).

**COROLLARY.** If \( \lim_{n \to \infty} \left| b_{n+1}/b_n \right| = k \), the continued fraction (1.1) will converge, at least in the wider sense, if \( k < 1 \), and will diverge if \( k > 1 \).

The proof is immediate.

**EXAMPLE.** A continued fraction with a circle as its region of convergence. Consider the continued fraction
\[
(1.4) \quad 1 + \frac{c_0}{1 + \frac{c_1 x}{1 + \frac{c_2 x^2}{1 + \cdots}}},
\]
where the \( c_n \) are complex nonzero numbers. If \( \lim_{n \to \infty} \left| c_{n+1}/c_n \right| = c \neq 0 \), it follows from the preceding corollary that the continued fraction (1.4) converges, at least in the wider sense, to a function analytic except possibly for a finite number of poles in every closed region wholly interior to the circle \( |x| = 1/c \), and diverges outside. Further, if \( \lim_{n \to \infty} \left| c_{n+1}/c_n \right| = 0 \), (1.4) converges to a function meromorphic throughout the finite plane.
2. A general theorem on convergence. Leighton and Wall [7] gave an example of a convergent continued fraction (1.1) where the elements $b_n$ were everywhere dense in the complex plane. The following theorem attacks the general question of convergence from a different point of view. We assume as usual that all $b_n \neq 0$.

**Theorem 2.** Let $m_0, m_1, m_2, \ldots$ be any sequence of positive integers such that $m_0 = 2, m_{n+1} - m_n \geq 2, (n = 0, 1, 2, \ldots)$. The numbers

\begin{equation}
(2.1) \quad b_{m_0}, b_{m_1}, b_{m_2}, \ldots
\end{equation}

can be chosen in such a fashion that with at most one value in the complex plane excluded from each of the numbers $b_n$ not contained in the set (2.1), the continued fraction (1.1) will converge.

Let $A_n/B_n$ represent the $n$th approximant of (1.1), where $A_n$ and $B_n$ are given by the usual recursion relations

\begin{equation}
(2.2) \quad A_n = A_{n-1} + b_n A_{n-2}, \quad B_n = B_{n-1} + b_n B_{n-2}, \quad n = 2, 3, 4, \ldots.
\end{equation}

By means of (2.2) write $A_j$ and $B_j, (j = 2, 3, \ldots, m_1 - 1)$, as

\begin{equation}
(2.2') \quad A_j = f_j A_1 + b_j A_0, \quad B_j = f_j B_1 + b_j B_0,
\end{equation}

where $f_j$ and $g_j$ are polynomials in the numbers $b_2, b_3, \ldots, b_{m_1 - 1}$ and do not depend on any other $b$'s. (Perron [8], p. 14, uses the symbol $A_{m-r, m_r}$ for $f_j$ and $B_{m-r, m_r}$ for $g_j$). Suppose the numbers $f_j$ are nonzero. It is clear that $|b_{m_0}| = |b_2|$ can be chosen so small that simultaneously

$$\left| \frac{A_j}{B_j} - \frac{A_1}{B_1} \right| < \frac{1}{2}, \quad j = 2, 3, \ldots, m_1 - 1.$$ 

Now write $A_k$ and $B_k, (k = m_1, m_1 + 1, \ldots, m_2 - 1)$, as

\begin{equation}
(2.2'') \quad A_k = f_k A_{m_1 - 1} + b_k g_k A_{m_1 - 2}, \quad B_k = f_k B_{m_1 - 1} + b_k g_k B_{m_1 - 2},
\end{equation}

where $f_k$ and $g_k$ are polynomials in $b_{m_1 + 1}, b_{m_1 + 2}, \ldots, b_{m_2 - 1}$ and do not depend on any $b$'s not in this set. Similarly, let us suppose for the moment that the numbers $f_k$ are never zero. The number $|b_{m_1}|$ can then be taken so small that

$$\left| \frac{A_k}{B_k} - \frac{A_{m_1 - 1}}{B_{m_1 - 1}} \right| < \frac{1}{2^k}, \quad k = m_1, m_1 + 1, \ldots, m_2 - 1.$$
Continue the process. With the assumption that \( f^t \) is never zero it is clear that \( b_m \) can be chosen so small that

\[
\left| \frac{A_t}{B_t} - \frac{A_{m-1}}{B_{m-1}} \right| < \frac{1}{2^{t+1}}, \quad t = m_r, m_r + 1, \ldots, m_{r+1} - 1.
\]

The continued fraction will thus converge.

It remains to assign conditions to the numbers \( b_n \) so that the numbers \( f^t \) will be different from zero. It is sufficient to exclude precisely one value in the finite complex plane from each \( b_n \) not in the set \( b_{m_1}, b_{m_2}, \ldots \). For, in the general case, it follows from (2.2) that

\[
f_r^{m_r} = 1, \quad f_r^{m_r+1} = 1 + b_{m_r+2},
\]

\[
f_r^{m_r+s} = f_r^{m_r+s-1} + b_{m_r+s} f_r^{m_r+s-2}, \quad s = 2, 3, \ldots, m_{r+1} - m_r - 1,
\]

where \( f_r^{m_r+s-1} \) is a polynomial in \( b_{m_r+2}, b_{m_r+3}, \ldots, b_{m_r+s-1} \) and depends on no other \( b \)'s. The value \(-1\) is first excluded from \( b_{m_r+2} \). It follows from (2.3) that one value may be excluded from each successive \( b \) in such a way that \( f^t \) is never zero. This completes the proof of the theorem.

**Bibliography**