SOME INVARIANTS UNDER MONOTONE TRANSFORMATIONS*

D. W. HALL† AND A. D. WALLACE

We assume that $S$ is a locally connected, connected, compact metric space and that $P$ is a property of point sets. For any two points $a$ and $b$ of $S$ we denote by $C(ab)$ (respectively $C_i(ab)$) a closed (closed irreducible) cutting of $S$ between the points $a$ and $b$. We consider the following properties:

- $\Delta_0(P)$. If $S$ is the sum of two continua, their product has property $P$.
- $\Delta_1(P)$. If $K$ is a subcontinuum of $S$ and $R$ is a component of $S - K$, then the boundary of $R$, $(F(R) = R - R)$, has property $P$.
- $\Delta_2(P)$. Each $C_i(ab)$ has property $P$.
- $\Delta_3(P)$. If $A$ and $B$ are disjoint closed sets containing the points $a$ and $b$, respectively, there is a $C(ab)$ disjoint from $A + B$ and having property $P$.

If $P$ is the property of being connected, the four properties $\Delta_i(P)$ are equivalent as shown by Kuratowski.† Indeed it may be seen that Kuratowski's proofs allow us to state the following theorem:

**Theorem 1.** For any property $P$ of point sets, $\Delta_i(P)$ implies $\Delta_{i+1}(P)$ for $i = 0, 1, 2$.

This result is the best possible in the sense that there is a property (that of being totally disconnected) for which no other implication holds.

The single-valued continuous transformation $T(S) = S'$ is said to be monotone if the inverse of every point is connected. It may be seen that the following statements are true:§

(i) **The inverse of every connected set is connected.**
(ii) **If the set $X$ separates $S$ between the inverses of the points $x$ and $y$, then $T(X)$ separates $S'$ between $x$ and $y.**

**Theorem 2.** If the property $P$ is invariant under monotone trans-

---

* Presented to the Society, October 29, 1938.
† National Research Fellow.
formations, then for each $i = 0, 1, 2, 3$, the property $\Delta_i(P)$ is invariant under the monotone transformation $T(S) = S'$.

**Proof.** (0) If $S' = L + M$, the summands being continua, then $S = L^{-1} + M^{-1}$ is a sum of continua. Hence the set $L^{-1} \cdot M^{-1}$ has property $P$ and $L \cdot M = T(L^{-1} \cdot M^{-1})$ then has property $P$.

(1) If $R$ is a component of $S' - K$, where $K$ is a continuum, then $R^{-1}$ is a component of the complement of the continuum $K^{-1}$. By assumption, $F(R^{-1})$ has property $P$. It follows that its image has property $P$. But we have $T(F(R^{-1})) = T(R^{-1} - R^{-1}) = T(R^{-1}) - R = F(R)$.

(2) Assume that $C$ is a $C_i(ab)$ in $S'$. From the continuity of $T$ it follows that $C^{-1}$ is a $C(pq)$ in $S$, $p$ and $q$ being any two points in the inverses of $a$ and $b$, respectively. Since the inverses of $a$ and $b$ are connected, there exists a cutting $K$ of $S$ between these two sets such that $K$ is a $C_i(xy)$, where $T(x) = a$ and $T(y) = b$; and further $K$ is a subset of $C^{-1}$. Thus $K$ has property $P$; hence $T(K)$ has. But $T(K) \subset C$, and $T(K)$ is a $C(ab)$. It follows that $T(K) = C$ and from this that $C$ has property $P$.

(3) Let $A$ and $B$ denote disjoint closed subsets of $S'$ containing $a$ and $b$. If $x$ and $y$ are points which map into $a$ and $b$, then by hypothesis there is a cutting $K$ of $S$ between $x$ and $y$ that is disjoint with $A^{-1}$ and $B^{-1}$ and has property $P$. Since, clearly, $K$ is a cutting of $S$ between the inverses of $a$ and $b$, it follows that $T(K)$ cuts $S'$ between $a$ and $b$, is disjoint with $A + B$, and has property $P$.

As an application we have the following known results:

**Theorem 3.** The property of a locally connected continuum to be a dendrite, a regular curve, or a rational curve is a monotone invariant.

To see this we take $P$ to be the property of being a point, a finite set of points, or a countable set of points and apply the invariance of $\Delta_S(P)$.

---

* If $X$ is a subset of $S'$, we denote by $X^{-1}$ the inverse of $X$.


‡ See the fourth footnote and references given there.