ON NON-BOUNDARY SETS*

A. D. WALLACE

The purpose of this note is largely methodological; namely, to complete the triology of dense, boundary, and nondense sets by adding non-boundary sets.

We adhere to the nomenclature of Kuratowski’s *Topologie†* except as noted. In particular, we suppose that $S$ is a nonvacuous space satisfying his axioms of closure, and we write $F(X) = \overline{X} \cdot CX$ and $X^0 = C\overline{CX}$ where $CX = S - X$. If the set $X$ has the property $P$, we write $X^p$ or $(X)^p$, and in the contrary case $X^{cp}$ or $(X)^{cp}$.

A set is **dense** if its closure is the space; **boundary** if its complement is dense; **nondense** if its closure is boundary; and finally, **non-boundary** if its complement is nondense. We designate the properties by $D$, $B$, $ND$, and $NB$, respectively.

**Theorem.** The following conditions are necessary and sufficient in order that a set be

I. Dense: The interior of its closure is the space; the boundary of its complement is the closure of its complement; its complement is a boundary set; its closure is a non-boundary set.

II. A boundary set: The closure of its interior is null; its boundary is its closure; its complement is dense; its interior is nondense.

III. Nondense: The interior of its closure is null; the boundary of its closure is its closure; its complement is a non-boundary set; its closure is a boundary set.

IV. A non-boundary set: The closure of its interior is the space; the boundary of the closure of its complement is the closure of its complement; its complement is nondense; its interior is dense.

We summarize this in the following table of equivalences. The Roman numerals correspond to the statements above, and each statement in a row is equivalent to every other statement in that row.

The proofs of these statements are as follows: Column 2 is a formulation of the definitions. In column 3 statement I 3 follows from I 2 since $S^0 = S$; II 2 is equivalent to $X^0 = 0$, which is clearly the same as II 3; III 3 is the complement of III 2; IV 3 is IV 2.

As to column 4, we have for I 4

\[
(X = S) \rightarrow (X \cdot CX = \overline{CX}) \rightarrow (F(CX) = \overline{CX}).
\]

* * * * *

* Presented to the Society, February 25, 1939.
Also $\overline{X} \cdot \overline{CX} = \overline{CX}$ is the same as saying that $\overline{CX}$ is a subset of $\overline{X}$. But $\overline{CX} \subset CX \subset \overline{CX} \subset \overline{X}$, or $\overline{CX} = 0$. The remaining statements in this column follow from this one using only the definitions.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$X^D$</td>
<td>$\overline{X} = S$</td>
<td>$\overline{X^0} = S$</td>
<td>$F(CX) = \overline{CX}$</td>
<td>$(CX)^B$</td>
<td>$(\overline{X})^{NB}$</td>
</tr>
<tr>
<td>II</td>
<td>$X^B$</td>
<td>$\overline{CX} = S$</td>
<td>$\overline{X^0} = 0$</td>
<td>$F(X) = \overline{X}$</td>
<td>$(CX)^D$</td>
<td>$(X^0)^{ND}$</td>
</tr>
<tr>
<td>III</td>
<td>$X^{ND}$</td>
<td>$\overline{CX} = S$</td>
<td>$\overline{X^0} = 0$</td>
<td>$F(X) = \overline{X}$</td>
<td>$(CX)^{NB}$</td>
<td>$(\overline{X})^B$</td>
</tr>
<tr>
<td>IV</td>
<td>$X^{NB}$</td>
<td>$\overline{CX} = S$</td>
<td>$\overline{X^0} = S$</td>
<td>$F(CX) = \overline{CX}$</td>
<td>$(CX)^{ND}$</td>
<td>$(X^0)^D$</td>
</tr>
</tbody>
</table>

Column 5 can be deduced from column 4 and the definitions. Statement IV 6 is the same as IV 2; III 6 is II 2 with $X$ replaced by $\overline{X}$; for II 6 we have

$$(X^0)^{ND} \equiv (\overline{CX}^0 = S) \equiv (\overline{CX} = S) \equiv X^B;$$

the proof of I 6 is similar.

We have the following theorem giving relations between the various properties:

**Theorem.** If a set is a non-boundary set, it is not a boundary set; if it is not a boundary set, it is not nondense; and if it is closed and not nondense, it is not a boundary set. If a set is a non-boundary set, it is dense; if it is dense, it is not nondense; if it is open and dense, it is a non-boundary set.

These results may be seen easily from the appended diagram, the arrow indicating the direction of implication. The proofs of the statements are as follows: $X^{NB} \rightarrow X^{CB}$ from IV 3 and II 3; $X^{CB} \rightarrow X^{CND}$, from II 2 and III 2; $X$ closed and not nondense $\rightarrow X^{CB}$ in the same manner; $X^{NB} \rightarrow X^D$, from I 2 and IV 3 since $X^0 \subset X$; $X$ open and dense $\rightarrow X^{NB}$, in a similar way; $X^D \rightarrow X^{CND}$, from I 3 and III 3.

The following results are typical of non-boundary sets.

A necessary and sufficient condition that $X$ be a non-boundary set is that $X$ be dense and the boundary of $X$ be nondense.
For we have
\[ X^D \rightarrow (C\overline{X} = 0) \rightarrow (\overline{C\overline{X}} = 0) \]
and
\[ F(X)^{ND} \rightarrow (C[\overline{C\overline{X}} + \overline{X^0}] = 0) \rightarrow (\overline{X^0} = S). \]

Conversely,
\[ (\overline{X^0} = S) \rightarrow (\overline{C\overline{X}} + \overline{X^0} = S) \rightarrow (F(X)^0 = 0) \rightarrow F(X)^{ND}. \]

The product of a countable collection of non-boundary sets is a residual set; that is, the complement of a set of the first category.

For if \( X \) is such a set, then \( CX \) is the sum of a countable collection of nondense sets by IV 5.

Every dense \( G_\delta \) is residual.*

In fact, if \( X = \prod X_n, X_n = X_n^0 \), then each \( X_n \) is a non-boundary set because
\[ S = \overline{X} \circ \overline{X_n} = \overline{X_n^0}. \]

In a complete metric space the product of a countable collection of non-boundary sets is not vacuous.

This is a classic theorem of R. Baire.

---

* See Kuratowski, loc. cit., p. 206, Theorem V 2. I owe this reference to a referee.