TOTALLY GEODESIC EINSTEIN SPACES*

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1. Introduction. An Einstein space† $E_m$ is defined as a Riemann space $V_m$ whose mean curvature $a$ is a constant at each point; that is,‡

$$R_{a\bar{a}} = - a g_{a\bar{a}}$$

where $R_{a\bar{a}}$ and $g_{a\bar{a}}$ are the Ricci and metric tensors of $V_m$, respectively. We suppose that the dimension $m$ of $E_m$ exceeds 3. For every surface is an $E_2$ and the only $E_3$'s are the spaces of constant curvature. In both these cases, the discussion which parallels that given in this note is obvious and trivial. Since $m > 3$, it is a well known consequence of (1.1) that $a$ is a constant throughout the space. In this note, we discuss the properties of an $E_m$ which admits families of totally geodesic subspaces which are also Einstein spaces. It is shown that this subject is closely related to the problems of finding (a) all Einstein spaces which may be imbedded as hypersurfaces of a space of constant curvature (b) Einstein spaces which are conformal to Einstein spaces. In a restricted sense,§ we also find the first fundamental form of $E_m$. It is assumed that the first fundamental forms of $E_m$ and of its subspaces which are discussed below are nonsingular although they may be indefinite.

2. Separable Einstein spaces. It has been shown by Bompiani|| that the necessary and sufficient condition that the subspaces $x^p = \text{const.}$ and the orthogonal subspaces $x^i = \text{const.}$ be totally geodesic in $V_m$ is that

$$(2.1) \quad g_{ij} = f_{ij}(x^k), \quad g_{pq} = h_{pq}(x^r), \quad g_{ip} = 0.$$  

When the first fundamental form of $V_m$ satisfies (2.1), it is called

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† We represent an $m$-dimensional Riemann space and Einstein space by $V_m$ and $E_m$, respectively.
‡ Throughout this note, $\alpha, \beta, \gamma, \delta; h, i, j, k; p, q, r; \lambda, \mu, \nu$ have the ranges $1, 2, \ldots, m; 1, 2, \ldots, n; n+1, n+2, \ldots, m; 1, 2, \ldots, n-1$, respectively. An index which appears twice in an expression is to be summed over the appropriate range. A free index of a tensor equation assumes each value of its range.
§ The first fundamental form of $E_m$ is obtained in a preferred coordinate system and depends upon the unknown first fundamental form of an arbitrary Einstein space.
separable, and the two forms \( f_i dx^i dx^i \) and \( h_{pq} dx^p dx^q \) are called its \textit{components}. It is clear that the components are the first fundamental forms of the totally geodesic \( V_n \)'s and \( V_{m-n} \)'s of \( V_m \) and that the \( V_n \)'s (as well as the \( V_{m-n} \)'s) are isometric.

We denote the Christoffel symbols of the first and second kind by \( [\alpha \beta, \gamma] \), \( \{\alpha \beta \gamma \} \) for \( V_m \); \( [ij, k] \), \( \{ijjk \} \) for \( V_n \), and \( [pq, r] \), \( \{pqr \} \) for \( V_{m-n} \). Then it follows from (2.1) that

\[
(2.2) \quad [ij,k] = [ij,k], \quad \{i \mid jk \} = \{i \mid jk \},
\]
\[
(2.3) \quad [pq,r] = [pq,r], \quad \{p \mid qr \} = \{p \mid qr \},
\]
\[
(2.4) \quad [\alpha \beta, \gamma] = 0, \quad \{\alpha \mid \beta \gamma \} = 0
\]

if \( \alpha, \beta, \gamma \) are not all in the same range. The Ricci tensor of \( V_m \) is defined as

\[
(2.5) \quad R_{\alpha \beta} = \frac{\partial^2 \log g^{1/2}}{\partial x^\alpha \partial x^\beta} - \frac{\partial}{\partial x^\gamma} \{\gamma \mid \alpha \beta \} + \{\gamma \mid \alpha \delta \} \{\delta \mid \gamma \beta \} - \{\gamma \mid \alpha \beta \} \frac{\partial \log g^{1/2}}{\partial x^\gamma}
\]

where \( g = |g_{\alpha \beta}| \).

We first suppose that \( n \geq 1 \) and \( m-n \geq 1 \). Then from (2.2), (2.3), (2.4), and (2.5),

\[
(2.6) \quad R_{ip} = 0, \quad R_{ij} = \overline{R}_{ij}, \quad R_{pq} = \overline{R}_{pq}
\]

where \( \overline{R}_{ij} \) and \( \overline{R}_{pq} \) are the Ricci tensors of \( V_n \) and \( V_{m-n} \), respectively. If \( V_m \) is an \( E_m \), it follows from (1.1), (2.1), and (2.6) that

\[
(2.7) \quad \overline{R}_{ij} = - a f_{ij}, \quad \overline{R}_{pq} = - a h_{pq}.
\]

Hence \( V_n \) and \( V_{m-n} \) are both Einstein spaces of the same mean curvature as \( E_m \). Conversely, by reversing the proof we find that (1.1) is a consequence of (2.1) and (2.7). This proves the following theorem:

\textbf{Theorem 2.1.} \textit{Let the first fundamental form of an Einstein space of dimensionality } \( m \geq 3 \) \textit{of mean curvature } \( a \) \textit{be separable into components whose dimensions exceed 1. Then each component is the first fundamental form of an Einstein space of mean curvature } \( a \). \textit{Conversely, only the first fundamental forms of Einstein spaces are separable in this manner.}

If \( m=4, n=2 \), it follows, from this theorem and the observation that a two-dimensional space of constant mean curvature has constant Riemann curvature, that each component is the first fundamental form of a space of constant curvature \( a \). This was first proved
by Kasner.* In recent papers,† we have shown that Einstein spaces which are proper hypersurfaces of any space of constant curvature are either spaces of constant curvature themselves or are separable. By a repeated application of Theorem 2.1, we may obtain an obvious generalization which applies to an \( E_m \) whose first fundamental form is separable into more than two components.

If \( n > 1 \) and \( m = n + 1 \), and if (2.1) is satisfied, the curves \( x^i = \text{const.} \) are geodesics of \( E_m \). In this case, equations (2.6) become

\[
R_{im} = 0, \quad R_{ij} = \bar{R}_{ij}, \quad R_{mm} = 0.
\]

As a consequence of (1.1), (2.1), and (2.8), we find that \( \bar{R}_{ij} = 0 \), \( a = 0 \), and conversely. We have proved the following theorem:

**Theorem 2.2.** A \( V_{n+1} \) which admits \( \infty^1 \) parallel totally geodesic \( E_n \)'s is an \( E_{n+1} \) if and only if the mean curvature of the \( E_n \)'s is zero. In this case, the mean curvature of \( E_{n+1} \) is also zero.

3. \( E_{n+1} \) with totally geodesic \( E_n \)'s. We suppose \( m = n + 1 \) and that \( E_{n+1} \) admits \( \infty^1 \) totally geodesic \( E_n \)'s which are not parallel. Since the first fundamental form of the necessarily isometric \( E_n \)'s is non-singular, it follows that the normals to the \( E_n \)'s in \( E_{n+1} \) are not null vectors.‡ Hence, in accordance with a slightly weaker form of the theorem of Bompiani quoted above,

\[
g_{ij} = f_{ij}(x^k), \quad g_{n+1,n+1} = \varepsilon H^2(x^i, y), \quad g_{i,n+1} = 0,
\]

where \( y = x^{n+1}, \varepsilon \) is \(+1\) or \(-1\), and \( f_{ij} dx^i dx^j \) is the first fundamental form of each \( E_n \). Since the hypersurfaces are not parallel, it follows that \( H(x^i, y) \) cannot be a function of \( y \) only but must involve the \( x^i \). Because of (3.1), we find that (2.2) holds and (2.4) is true if one of \( \alpha, \beta, \gamma \) is \( n + 1 \) and the other two lie in the range \( 1, 2, \cdots, n \). Also

\[
\left\{ n + 1 \middle| i \ n + 1 \right\} = \frac{\partial \log H}{\partial x^i}, \quad \left\{ n + 1 \middle| n + 1 \ n + 1 \right\} = \frac{\partial \log H}{\partial y},
\]

(3.2)

\[
\left\{ i \middle| n + 1 \ n + 1 \right\} = -eH^2 \frac{\partial \log H}{\partial x^i} \cdot g^{ij},
\]


where \( g^{ij} \) are the contravariant components of \( g_{ij} \). From (2.2), (2.4), (2.5), and (3.2),

\[
R_{n+1,n+1} = - \frac{\partial}{\partial x^i} \{ i \mid n + 1, n + 1 \} + \{ n + 1 \mid n + 1, i \} \{ i \mid n + 1, n + 1 \} - \{ i \mid n + 1, n + 1 \} \frac{\partial \log f^{1/2}}{\partial x^i},
\]

(3.3)

\[
R_{i,n+1} = 0,
\]

(3.4)

\[
R_{ij} = \overline{R}_{ij} + (\log H)_{,ij} + (\log H)_{,i}(\log H)_{,j},
\]

where the comma denotes covariant differentiation with respect to the form \( f_{ij}dx^idx^j \), and \( f = |f_{ij}| \). Since \( E_{n+1} \) and \( E_n \) are Einstein spaces, (1.1) and

\[
(3.6) \quad \overline{R}_{ij} = -bf_{ij}
\]

are true.

It follows from (1.1), (3.1), and (3.6) that (3.5) becomes

\[
(3.7) \quad H_{,ij} = cHf_{ij}
\]

where

\[
(3.8) \quad c = b - a.
\]

The integrability conditions of (3.7) are

\[
H_{,ijk} - H_{,ikj} = H_{,h} \overline{R}_{ijk}^h
\]

which become, by virtue of (3.7),

\[
H_{,h} \overline{R}_{ijk}^h = c(H_{,h}f_{ij} - H_{,h}f_{ik}).
\]

The tensor \( \overline{R}_{ijk}^h \) is the Riemann curvature tensor of \( E_n \). If we multiply this equation by the contravariant components \( f^{ij} \) of the metric tensor of \( E_n \) and sum for \( i, j \), we find after using (3.6) that

\[
[c(n - 1) + b]H_{,k} = 0.
\]

Since \( H_{,k} \) cannot be zero for every value of \( k \), the last equation and (3.8) show that

\[
(3.9) \quad nb = (n - 1)a
\]

is a necessary condition that (3.7) have a solution.

Since \( \partial \log (f)^{1/2}/\partial x^i = \{ k \mid ki \} \), after using (3.2), equation (3.3) becomes
\[ R_{n+1,n+1} = eH \left[ \frac{\partial}{\partial x^i} (f^{ij}H_{,i} + f^{ij}H \{ k | i \} ) \right] \]

or

\[ R_{n+1,n+1} = eH \Delta_2 H \]

where \( \Delta_2 H = f^{ij}H_{,ij} \). From (3.7), (3.8), (3.9), and (3.10),

\[ R_{n+1,n+1} = -aeH^2. \]

According to (3.1), this equation and (3.4) both obey (1.1). Hence \( E_{n+1} \) will admit one nonparallel totally geodesic \( E_n \)'s if and only if a solution of (3.7) exists.

Since \( H(x^i, y) \) is not independent of the \( x^i \), we may choose coordinates \( x^i \) so that \( H = x^n \) for some fixed value of \( y \) and such that \( f_{\alpha \lambda} = 0 \). Then (3.7) becomes

\[- \{ n | ij \} = ex^nf_{ij}.\]

Now Brinkmann* has shown that \( E_n \) admits a solution of these equations if and only if its metric tensor satisfies

\[ f_{nn} = (ex^n + d)^{-1}, \]

\[ f_{\lambda \mu} = (ex^n + d)F_{\lambda \mu}(x^n), \quad f_{n\lambda} = 0, \quad d \text{ constant}, \]

and the form \( F_{\lambda \mu}(x^n)dx^\lambda dx^\mu \) is the first fundamental form of an \( E_{n-1} \). According to Brinkmann, (3.11) is the necessary and sufficient condition that \( E_n \) be conformal to another Einstein space by means of a transformation \( ds = \sigma ds \) with \( \Delta_1 \sigma \neq 0 \) where \( \Delta_1 \sigma = f^{ij}g_{ij} \). If we suppose \( H = x^n \) for all values of \( y \), using (3.1) we may write the first fundamental form of one \( E_{n+1} \) which satisfies the conditions of the problem as

\[ ds^2 = f_{ij}dx^idx^j + ex^n^2dx^{n+1}. \]

where the \( f_{ij} \) satisfy (3.11). This proves the following theorem:

**Theorem 3.1.** A one-parameter family of isometric \( E_n \)'s may be imbedded as nonparallel totally geodesic hypersurfaces of an \( E_{n+1} \) if and only if each \( E_n \) may be mapped conformally on another Einstein space by means of a function \( \sigma \) with \( \Delta_1 \sigma \neq 0 \). If \( a \) and \( b \) are the mean curvatures of \( E_{n+1} \) and \( E_n \), respectively, then \( nb = (n-1)a \).

We now briefly consider the conditions under which the \( E_n \) de-

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fined by (3.11) admits solutions $H(x^n, y)$ of the equations (3.7) other than $H = x^n$. By methods similar to those hitherto employed, we find that the most general solution for $H$ of the form $H = H(x^n, y)$ is

$$
H = \alpha(y) \cdot x^n + \beta(y), \quad c = 0,
$$

or

$$
H = \alpha(y) \cdot x^n, \quad c \neq 0,
$$

where $\alpha(y)$ and $\beta(y)$ are arbitrary functions of $y$. We note that the $E_{n+1}$ obtained by using the $H$ defined by (3.14) coincides with (3.12).

It can be shown that solutions for $H$ which involve some of the $x^r$ do not exist unless the $E_n$ defined by (3.11) may be mapped conformally on another Einstein space in more than one way. Hence, if this is not the case, the $E_n$'s may only be imbedded in the unique $E_{n+1}$ defined by (3.12) if $c \neq 0$ and only in the $E_{n+1}$'s defined by (3.1), (3.11), and (3.13) if $c = 0$. In this last case, $a = b = 0$.

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CONCERNING THE BOUNDARY OF A COMPLEMENTARY DOMAIN OF A CONTINUOUS CURVE*

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Much study by various investigators has been given to the nature of the boundary of a complementary domain of a locally compact continuous curve in the plane and in certain other spaces.† It is the purpose of this paper to continue this investigation in less restricted spaces which satisfy the Jordan curve theorem and to establish certain results (from which many of the known results follow immediately) in such a way as to bring out what is essential for their validity.

It is first necessary to establish the following lemma.

**Lemma A.** *If a locally compact nondegenerate continuous curve $M$ in a complete Moore space contains no simple triod, then $M$ is a simple continuous curve."*‡

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† See the bibliography and Chapter 4 of R. L. Moore's *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, New York, 1932. Hereinafter, this book will be referred to as *Foundations*, and the reader is referred to it for many theorems and the definition of certain terms and phrases used in this paper.
‡ A *complete Moore space* is a space satisfying Axioms 0 and 1 of *Foundations*. A *simple continuous curve* is either a simple continuous arc, a simple closed curve, an open curve, or a ray.