A CHARACTERIZATION OF DEDEKIND STRUCTURES*

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If \( \Sigma \) is a Dedekind structure, then for any two elements \( A \) and \( B \) of \( \Sigma \), the quotient structures \([A, B]/A\) and \( B/(A, B)\) are isomorphic. (Dedekind [2], Ore [3].) I prove here a converse result.

**Theorem.** Let \( \Sigma \) be a structure in which for every pair of elements \( A \) and \( B \), the quotient structures \([A, B]/A\) and \( B/(A, B)\) are isomorphic. Then if either the ascending or descending chain condition holds in \( \Sigma \), the structure is Dedekindian.

This result is comparatively trivial if both the ascending and descending chain conditions hold. That some sort of chain condition is necessary may be seen by a simple example. Consider a structure \( \Sigma \) with an all element \( O_0 \) and a unit element \( E_0 \) built up out of three ordered structures \( \Sigma_1 \), \( \Sigma_2 \), \( \Sigma_3 \) meeting only at \( O_0 \) and \( E_0 \), so that if \( S_u \in \Sigma_u \), then

\[
(S_u, S_v) = E_0, \quad [S_u, S_v] = O_0
\]

for \( u, v = 1, 2, 3, u \neq v \). Then if each \( \Sigma_i \) is a series of the type of the real numbers in the closed interval \( 0, 1 \), the quotient structures of any pair \([S_u, S_v]/S_u, S_v/(S_u, S_v)\) are obviously isomorphic. But \( \Sigma \) is clearly non-Dedekindian.

The theorem is of some interest in view of the generalizations Ore has given of his decomposition theorems in Ore [4].

It suffices to prove the result under the hypothesis that the descending chain axiom holds in \( \Sigma \) (Ore [3, p. 410]). We formulate this axiom as follows:

(\( \beta \)) *If for any two elements \( A \) and \( B \) of \( \Sigma \),

\[
A \triangleright X_1 \triangleright X_2 \triangleright X_3 \triangleright \cdots \triangleright B
\]

for an infinity of \( X_i \) in \( \Sigma \), all the \( X_i \) are equal from a certain point on.*

Our proof rests upon several lemmas which we collect here.

**Lemma 1.** (Dedekind [2].) \( \Sigma \) is a Dedekind structure if and only if \( \Sigma \) contains no substructure \( \Sigma_0 \) of order five which is non-Dedekindian.

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\( \dagger \) We use the notation and terminology of Ore's fundamental paper, Ore [3], with the following two exceptions. (i) We write \( A \triangleright B \), \( B \triangleleft A \) for Ore's \( A \geq B \), \( B \leq A \). (ii) If \( A \) is prime over \( B \) (Ore [3, p. 411]), we shall say "\( A \) covers \( B \)" or "\( B \) is covered by \( A \)" (Birkhoff [1]) and write \( A > B \) or \( B < A \).
The type of substructure in question is well known; its diagram is given in the figure. Since we utilize such substructures frequently in our proof, we shall introduce the notation \( \{D, A, B, C, M\} \) for \( \Sigma_0 \), writing the all element \( D \) and unit element \( M \) in the first and last places in the symbol while the elements \( A \) and \( B \) where \( A \triangleright B \) occupy the second and third places.

**Lemma 2.** (Ore [3].) If (β) holds in the structure \( \Sigma \), then every set of elements of \( \Sigma \) which divide a fixed element \( A \) contains at least one minimal element dividing no other element of the set.

**Lemma 3.** If (β) holds in the structure \( \Sigma \), then for any two distinct elements \( A \) and \( C \) of \( \Sigma \) such that \( C \) divides \( A \), there exists an element \( B \) such that \( C \) divides \( B \) and \( B \) covers \( A \).

For we need only pick a minimal element in the subset of all elements \( X \) such that \( C \triangleright X \triangleright A, X \neq A \).

The following lemma is obvious:

**Lemma 4.** Let \( \Sigma \) be a structure in which

(ε)

\[ [A, B]/A \simeq B/(A, B) \]

for every \( A, B \) of \( \Sigma \). Then \( [A, B] \) covers \( A \) if and only if \( B \) covers \( (A, B) \).

**Lemma 5.** Let \( \Sigma \) be a structure in which (ε) holds. Then if \( A \) covers \( B \) and \( M \) is any other element of \( \Sigma \), either \([M, A]\) equals \([M, B]\) or \([M, A]\) covers \([M, B]\).

For clearly \([M, A] \triangleright [M, B]\). Since \( A \triangleright (A, [M, B]) \triangleright B \) and \( A \triangleright B \), either \((A, [M, B]) = A \) or \((A, [M, B]) = B \). If \((A, [M, B]) = A \), then \([M, B] \triangleright A \triangleright [M, A] \), so that \([M, B] = [M, A] \). If \((A, [M, B]) = B \), then \( A \triangleright (A, [M, B]) \). Hence by Lemma 4, \([A, [M, B]] \triangleright [M, B]\). But since \( A \triangleright B \),

\[ [A, [M, B]] = [M, A] \].

Our final lemma is the dual of Lemma 5.
**Lemma 6.** Let \( \Sigma \) be a structure in which (e) holds. Then if \( A \) covers \( B \) and \( M \) is any other element of \( \Sigma \), either \( (M, A) \) equals \( (M, B) \) or \( (M, A) \) covers \( (M, B) \).

We shall prove our theorem indirectly. Assume that conditions (\( \beta \)) and (e) hold in the structure \( \Sigma \), but that \( \Sigma \) is non-Dedekindian. Then by Lemma 1, \( \Sigma \) contains a non-Dedekindian substructure

\[ \Sigma_0 = \{ D, A, B, C, M \} \]

of order five.*

We may assume that \( A \) covers \( B \). For by Lemma 3, there exists an element \( N \) of \( \Sigma \) such that \( A \succ N, N \succ B \). Thus

\[ [A, C] \succ [N, C] \succ [B, C], \quad (A, C) \succ (N, C) \succ (B, C); \]

that is, \( [N, C] = D, (N, C) = M \). Hence \( \{ D, N, B, C, M \} \) is a non-Dedekindian substructure where \( N \succ B \).

We assume henceforth that \( A \) covers \( B \). Since \( [A, C] = D, (A, C) = M \), and \( [B, C] = D, (B, C) = M, D/C \cong A/M \), and \( D/C \cong B/M \) by (e). Hence \( A/M \cong B/M \). But \( B \) lies in \( A/M \) and \( A \succ B \). Since \( A \) corresponds to \( B \) under the isomorphism, there exists an element in \( B/M \) covered by \( B \). Denote it by \( B_1 \). Then

(1) \[ B \succ B_1 \succ M. \]

Since \( B \succ B_1 \succ M, (B, C) \succ (B_1, C) \succ (M, C) \) or \( (B_1, C) = M \). Consider next the union \( D_1 = [B_1, C] \). Since \( B \succ B_1 \), by Lemma 5 either \( [B, C] = [B_1, C] \) or \( [B, C] \succ [B_1, C] \); that is, either \( D = D_1 \) or \( D \succ D_1 \).

If \( D = D_1 \), then on writing \( A_1 \) for \( B \), we obtain a non-Dedekindian substructure \( \{ D_1, A_1, B_1, C, M \} \) in which \( A_1 \succ B_1 \).

Now assume that \( D \succ D_1 \). Clearly \( [A, D_1] = [B, D_1] = D \). Consider the crosscut \( (B, D_1) \). Since \( B \succ B_1 \), by Lemma 6, either \( (B, D_1) = (B_1, D_1) \) or \( (B, D_1) \succ (B_1, D_1) \). That is, since \( B \succ (B, D_1) \) and \( D_1 \succ B_1 \), either \( (B, D_1) = B_1 \) or \( (B, D_1) \succ B_1 \). We must have \( (B, D_1) = B_1 \).

For if \( (B, D_1) \succ B_1 \), then \( D_1 \succ B \). Since \( D_1 \succ C \), we would have \( D_1 \succ [B, C], D_1 = D \), contrary to the assumption \( D \succ D_1 \).

Consider next the crosscut \( A_1 = (A, D_1) \). Since \( A \succ B \), by Lemma 5 either \( (A, D_1) = (B, D_1) \) or \( (A, D_1) \succ (B, D_1) \); that is, either \( A_1 = B_1 \) or \( A_1 \succ B_1 \). We must have \( A_1 \succ B_1 \). For if \( A_1 = B_1 \), then \( \{ D, A, B, D_1, B_1 \} \) is a non-Dedekindian substructure. But since \( [A, D_1] = D \) and \( (A, D_1) = B_1 \), by (e) \( A/B_1 \cong D/D_1 \). This isomorphism is impossible, for \( A \succ B \succ B_1 \) while \( D \succ D_1 \).

Finally, since \( A \succ A_1 \succ C \) and \( B \succ B_1 \succ C \), \( (A_1, C) = (B_1, C) = M \)

* The reader will find a structure diagram helpful in following the argument.
while \([A_1, C] = [B_1, C] = D_1\). Thus \(\{D_1, A_1, B_1, C, M\}\) is a non-Dedekindian substructure of \(\Sigma\) in which \(A_1 > B_1\).

We now replace \(\Sigma_0\) in either case by \(\Sigma_1 = \{D_1, A_1, B_1, C, M\}\) and obtain a non-Dedekindian substructure \(\Sigma_2 = \{D_2, A_2, B_2, C, M\}\) where \(A_2 > B_2\) and

\[
B_1 > B_2 > M.
\]

On repeating this reasoning, and combining (1), (2), \ldots we obtain a chain

\[
B > B_1 > B_2 > B_3 > \cdots > M
\]

of indefinite length in which all \(B_i\) are distinct, contradicting (\(\beta\)).

**References**


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