SPACES OF UNCOUNTABLY MANY DIMENSIONS*

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Riemann in his Habilitations Schrift of 1854 suggested the notion of \( n \)-dimensional space (where \( n \) is a natural number) as an extension of the notion of three-dimensional euclidean space. Hilbert extended the notion still further by defining a space of a countably infinite number of dimensions. Fréchet\(^{†}\) in 1908 defined two other spaces of countably many dimensions, which he called \( D_\omega \) and \( E_\omega \). Tychonoff\(^{‡}\) in 1930 defined a series of spaces of an unlimited number of dimensions and established several of their properties. The present paper undertakes, by generalizing the notions of spaces \( D_\omega \) and \( E_\omega \), to define spaces \( D_\alpha \) and \( E_\alpha \), respectively, for each cardinal number \( \aleph_\alpha \) representing the number of dimensions. It is shown that every metric space is homeomorphic with a subset of some space \( D_\alpha \). Certain properties of Tychonoff's spaces, here called spaces \( T_\alpha \), are also presented.

1. Spaces \( D_\alpha \). For each initial ordinal number \( \omega_\alpha \) the set of all points of space \( D_\alpha \) is the set of all type \( \omega_\alpha \) sequences \( [x_i]_{\omega_\alpha} \) of real numbers \( x_i \), such that \( 0 < x_i < 1 \). Suppose that \( [P_i]_{\omega_\alpha} \) is a type \( \omega_\alpha \) sequence of points of space \( D_\alpha \) such that for each \( i \), \( P_i = [x_{i,i}]_{\omega_\alpha} \); and suppose that \( P = [y_j]_{\omega_\alpha} \) is a point of space \( D_\alpha \). The sequence \( [P_i]_{\omega_\alpha} \) is said to converge to the point \( P \) if and only if it is true that if \( \epsilon \) is a positive number, there exists a positive integer \( N_\epsilon \) such that if \( i > N_\epsilon \), then \( |y_j - x_{i,i}| < \epsilon \) for every value of \( j < \omega_\alpha \). The point \( P \) is said to be the sequential limit point of \( [P_i]_{\omega_\alpha} \). A point \( P \) is said to be a limit point of a point set \( M \), provided there exists a sequence of distinct points of \( M \) which converges to \( P \). Thus space \( D_0 \) is equivalent to space \( D_\omega \) of Fréchet. For each two distinct points \( A = [x_i]_{\omega_\alpha} \) and \( B = [z_i]_{\omega_\alpha} \) of \( D_\alpha \) such that for each \( i \), \( x_i \neq z_i \), let \( D_{(A,B)}^\alpha \), also referred to as segment \( AB \), denote the set of all points \( P = [y_i]_{\omega_\alpha} \) such that \( x_i < y_i < z_i \) or \( z_i < y_i < x_i \). If there exist constants \( h \) and \( k \), \((h < k)\), such that for each \( i \), \( x_i = h \) and \( y_i = k \), the notation \( D_{(h,k)}^\alpha \) is used. It is evident that \( D_{(h,k)}^\alpha \) is homeomorphic with \( D_\alpha \) for every value of \( \alpha \), \( h \), and \( k \); but there exist points \( A \) and \( B \) such that \( D_{(A,B)}^\alpha \) is not homeomorphic with \( D_\alpha \).

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Theorem 1.1. Space $D^\alpha$ is metric, for every value of $\alpha$.

Suppose that $A = [x_i]^{\omega_\alpha}$ and $B = [y_i]^{\omega_\alpha}$ are points of $D^\alpha$. Let the distance from $A$ to $B$, $d(A, B)$, be the least upper bound of the numbers of the sequence $[|y_i - x_i|]^{\omega_\alpha}$. This definition can be shown to satisfy the conditions of Fréchet for metric spaces, as in the case of space $D^\omega$.

Theorem 1.2. In space $D^\alpha$ there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point.

Let $M$ denote the set of all points $P$ such that each of the coordinates of $P$ is either $1/4$ or $1/2$. Then $M$ is of power $2^{\aleph_\alpha}$. If $A$ and $B$ are distinct points of $M$, then $d(A, B) = 1/4$. Hence no point of $M$ is a limit point of $M$. But $M$ is closed since any limit point of $M$ has each of its coordinates either $1/4$ or $1/2$. Hence $M$ has no limit point.

Theorem 1.3. Space $D^\alpha$ is not $\mathfrak{N}_2$-separable* if $\mathfrak{N}_2 < 2^{\aleph_\alpha}$.

Suppose the theorem is false. There exists a cardinal number $\mathfrak{N}_\beta < 2^{\aleph_\alpha}$ such that there exists a subset $K$ of power $\mathfrak{N}_\beta$, everywhere dense in $D^\alpha$. Hence $D^\alpha$ is $\mathfrak{N}_\beta$-completely separable.* Let $G$ denote a collection of power $\mathfrak{N}_\beta$ of domains with respect to which $D^\alpha$ is $\mathfrak{N}_\beta$-completely separable. By Theorem 1.2 there exists a point set of power $2^{\aleph_\alpha}$ which has no limit point. For every point $P$ of $M$ there exists a domain of $G$ which contains $P$ and no other point of $M$; and this involves a contradiction.

Theorem 1.4. If $M$ is an $\mathfrak{N}_\omega$-separable metric space, then $M$ is homeomorphic with a subset of space $D^\alpha$.

This theorem is a generalization of a theorem of Urysohn,† and the proof is a generalization of that of Urysohn.

There exists a distance function $d(A, B)$ such that if $A$ and $B$ are distinct points of $M$, then $0 < d(A, B) < 1$. Let $K$ denote a subset of $M$, of power $\mathfrak{N}_\alpha$, everywhere dense in $M$. Let $[K_i]^{\omega_\alpha}$ denote a type $\omega_\alpha$ sequence of all the elements of $K$. For each point $P$ of $M$ let

$$\phi(P) = [1/2 + 1/2d(P, K_i)]^{\omega_\alpha}.$$ 

The correspondence is one-to-one. To show this, it is sufficient to

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show that if \( A \) and \( B \) are distinct points, \( \phi(A) \neq \phi(B) \). Suppose that \( A \) and \( B \) are distinct points. There exists a point \( K_i \) of \( K \) such that

\[
d(A, K_i) < 1/3 d(A, B).
\]

But

\[
d(A, K_i) + d(K_i, B) \geq d(A, B).
\]

Hence \( d(A, K_i) < d(K_i, B) \), and hence \( \phi(A) \neq \phi(B) \).

The correspondence is also bicontinuous. Suppose that \( \epsilon \) is a positive number and \( P \) and \( Q \) are points of \( M \) such that \( d(P, Q) < \epsilon \). Let \( \phi(P) = [x_i]^{p_\alpha} \) and \( \phi(Q) = [y_i]^{p_\alpha} \). Suppose further that \( K_i \) is an element of \( [K_i]^{p_\alpha} \). Then

\[
d(P, K_i) + d(P, Q) \geq d(Q, K_i), \quad d(Q, K_i) + d(P, Q) \geq d(P, K_i).
\]

Since \( d(P, Q) < \epsilon \), it follows that

\[
| d(Q, K_i) - d(P, K_i) | < \epsilon
\]

and hence

\[
| x_i - y_i | < \epsilon.
\]

Since this relation holds for every value of \( i \), it follows that if \( P \) is a limit point of a point set \( H \), then \( \phi(P) \) is a limit point of \( \phi(H) \).

Suppose now that \( \delta \) is a positive number and \( P \) and \( Q \) are points of \( M \) such that \( d(P, Q) > \delta \). Let \( \phi(P) = [x_i]^{p_\alpha} \), \( \phi(Q) = [y_i]^{p_\alpha} \), and \( d(P, Q) = \delta + \eta \). There exists an ordinal number \( x \) such that \( d(Q, K_x) < \eta/2 \). It follows that

\[
| d(P, K_x) - d(P, Q) | < \eta/2.
\]

Hence

\[
d(P, K_x) > \delta + \eta/2,
\]

hence

\[
d(P, K_x) - d(Q, K_x) > \delta,
\]

and hence

\[
| x_x - y_x | > \delta/2.
\]

Therefore if \( P \) is not a limit point of a point set \( H \), then \( \phi(P) \) is not a limit point of \( \phi(H) \).

**Note.** Since space \( D^\alpha \) is not \( \aleph_\alpha \)-separable for any cardinal number \( \aleph_\alpha < 2^{\aleph_\alpha} \), the power of the space, it thus appears that there is a relation between the dimensionality of a metric space and its type of
separability. The type of separability does not determine the dimensionality, but it fixes an upper bound to it.

Theorem 1.5. In order that a space $S$ be metric, it is necessary and sufficient that there exist a cardinal number $\aleph_\alpha$ such that $S$ is homeomorphic with a subset of space $D^\alpha$.

This follows with the help of Theorems 1.3, 1.4, and the fact that every metric space is $\aleph_\alpha$-separable for some cardinal number $\aleph_\alpha$.

Definition. A collection $G$ of real-valued functions defined over a point set $M$ is said to be equi-continuous provided it is true that if $P$ is a point of $M$ and $\epsilon$ is a positive number, there exists an open subset $D$ of $M$ containing $P$ such that if $f$ is a function of $G$, the oscillation of $f$ on $D$ is less than $\epsilon$.

Theorem 1.6. In order that a Hausdorff topological space $S$ be homeomorphic with a subset of space $D^\alpha$, it is necessary and sufficient that there exist an equi-continuous collection $G$, of power $\aleph_\alpha$, of functions defined over $S$, whose values are real numbers between 0 and 1, such that if $P$ is a point and $H$ is a closed point set not containing $P$, there exists a positive number $\delta$ such that if $Q$ is a point of $H$, there exists a function $f$ of $G$ such that

$$|f(Q) - f(P)| > \delta.$$ 

The condition is sufficient. Let $\gamma = [f_i]$ denote a type $\omega_\alpha$ sequence of the elements of $G$. For each point $P$, let $\phi(P) = [f_i(P)]$, then $\phi$ represents a one-to-one continuous transformation of $S$ into a subset of space $D^\alpha$.

The condition is also necessary. Suppose that $\phi(P)$ is a one-to-one continuous transformation of $S$ into a subset of space $D^\alpha$. For each ordinal number $\nu < \omega_\alpha$ and each point $P$, where $\phi(P) = [x_i]$, let $f_{\nu}(P) = x_\nu$. Let $G$ denote the collection of all functions $f_\nu$ so defined. Then $G$ is the required collection.

2. Spaces $E^\alpha$. If the conditions, employed in defining spaces $D^\alpha$, for the convergence of a sequence of points are made less restrictive so that it is no longer required that the sequences of coordinates converge uniformly, spaces $E^\alpha$ are defined. Using the notation of §1, the sequence of points $[P_i]$, is said to converge to the point $P$ if and only if it is true that if $j$ is an ordinal number less than $\omega_\alpha$ and $\epsilon$ is a positive number, then there exists a positive integer $N_{\epsilon,j}$ such that if $i > N_{\epsilon,j}$ then $|y_i - x_{i,j}| < \epsilon$. Space $E^\alpha$ is thus equivalent to space $E^\alpha$ of Fréchet. Every segment $E^\alpha_{(A,B)}$ is homeomorphic with $E^\alpha$. 
**Theorem 2.1.** There exists in space $E^1$ a compact, non-separable point set.

Let $H$ denote the set of all points $P = [x_i]_{i=1}^{\omega_1}$ such that for each $j$, $1/4 \leq x_j \leq 1/2$ and such that there exists an ordinal number $\delta_P < \omega_1$ such that if $j > \delta_P$, then $x_j = 1/4$. This set is compact. Suppose that $[P_i]_{i=1}^{\omega_1}$ is a type $\omega_1$ sequence of points $P_i = [x_{i,j}]_{i=1}^{\omega_1}$ of $H$; this sequence has a convergent subsequence. For each point $P_i$ there exists an ordinal number $\delta_i$ such that if $j > \delta_i$ then $x_{i,j} = 1/4$. The sequence of elements $\delta_i$ is a type $\omega_0$ sequence; let $\mu$ denote the smallest ordinal number greater than all the elements of this sequence. Then $\mu < \omega_1$.

Let $f$ denote a one-to-one correspondence between the ordinal numbers less than $\omega_0$ and the ordinal numbers less than $\mu$. Let $K$ denote the set of all points $Q = [y_i]_{i=1}^{\omega_1}$ such that if $i > \mu$ then $y_i = 1/4$. For each point $Q = [y_i]_{i=1}^{\omega_1}$ of $K$ let $\phi(Q) = [y_i]_{i=\mu}^{\omega_1}$. Then $\phi$ represents a one-to-one bicontinuous transformation of the points of $K$ into a subset of space $E^0$ such that if $z_h$, $(h < \omega_0)$, is a coordinate of a point of $\phi(H)$, then $1/4 \leq z_h \leq 1/2$. But in space $E^0$ the set $M$ of all points whose coordinates are greater than or equal to $1/4$ and less than or equal to $1/2$ is closed and compact. There exists a subsequence of $[\phi(P_i)]_{i=1}^{\omega_1}$, namely $[\phi(P_{i_n})]_{i_n=1}^{\omega_1}$, which converges to a point $T$ of $M$. Hence $[P_{i_n}]_{i_n=1}^{\omega_1}$ converges to $\phi^{-1}(T)$.

The point set $H$ is not separable. Suppose that it is separable. Then there exists a countable everywhere dense subset $M$. There exists an ordinal number $\xi < \omega_1$ such that if $P = [x_i]_{i=1}^{\omega_1}$ is a point of $M$, then if $i > \xi$, $x_i = 1/4$. But no point which fails to satisfy this condition is a limit point of $M$, and since there are uncountably many such points, this leads to a contradiction.

**Theorem 2.2.** Space $E^\alpha$ is not metric for $\alpha \geq 1$.

By Theorem 2.1, $E^1$ contains a compact, non-separable subset $H$. Hence if $\alpha \geq 1$, $E^\alpha$ contains a compact, non-separable subset. But every compact subset of a metric space is separable. Hence $E^\alpha$ is not metric for $\alpha \geq 1$.

**Theorem 2.3.** If $S$ is a normal, $\aleph_\alpha$-completely separable, topological space (of Hausdorff), then $S$ is homeomorphic with a subset of the space $E^\alpha$.

This theorem is a generalization of a theorem of Urysohn* to the effect that every normal, completely separable, Hausdorff topological

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space is homeomorphic with a subset of Hilbert space. Tychonoff has established a similar theorem for spaces $T^\alpha$.

**Theorem 2.4.** If $S$ is a normal topological space (of Hausdorff), there exists a cardinal number $\aleph_\alpha$ such that $S$ is homeomorphic with a subset of space $E^\alpha$.

This follows with the help of Theorem 2.3 and the fact that $S$ is $\aleph_\alpha$-completely separable for some value of $\aleph_\alpha$.

3. Spaces $T^\alpha$. By a further extension of the notion of limit point, Tychonoff's spaces, here called spaces $T^\alpha$, are produced. In order to define limit point in these spaces, the notion of region is introduced. Suppose that $G_\alpha$ denotes the collection of all finite sets of ordinal numbers $\nu < \omega_\alpha$. Then $G_\alpha$ is of power $\aleph_\alpha$. Let $\Pi_\alpha = [h_\alpha]^{\aleph_\alpha}$ denote a type $\omega_\alpha$ sequence of all the elements of $G_\alpha$. If $\nu$ is an ordinal number less than $\omega_\alpha$, $\nu$ a positive real number, and $A = [a_\nu]^{\aleph_\alpha}$ a point, let $R(A, r, \nu)$ denote the set of all points $P = [x_\nu]^{\aleph_\alpha}$ such that if $i$ is an element of $h_\nu$, then $(a_i - r) < x_i < (a_i + r)$. Then $R(A, r, \nu)$ is called a region. The region $R(A, r, \nu)$ is said to have center at $A$, constrictions at $h_\nu$, and radius $r$. For all values of $A$, $\nu$, and $r$, $R(A, r, \nu)$ is an open set in space $E^\alpha$. If we now adopt the definition that a point $P$ is a limit point of a point set $M$ if and only if every region that contains $P$ contains a point of $M$ distinct from $P$, we find that there exist point sets $M$ and points $P$ such that $P$ is a limit point of $M$ in this sense, but not in the sense of spaces $E^\alpha$. This new definition thus constitutes an extension of the notion of limit point and the resulting spaces are designated by $T^\alpha$. For example, consider the point set $M$ in space $T^1$ consisting of all points $Q = [x_{i, j}]^{\aleph_1}$, where $x_{i, j} = 1/4$ for $j < i$ and $x_{i, j} = 1/2$ for $j \geq i$. The point set $M$ does not have the point $P = [y_1]^{\aleph_1}$ (where for each $j$, $y_j = 1/4$) as a limit point in space $E^1$. However, $P$ is a limit point of $M$ in space $T^1$.

**Theorem 3.1.** For each cardinal number $\aleph_\alpha$, space $T^\alpha$ is a distributive space $\Gamma_\aleph_\alpha$.

A space is said to be a space $\Gamma_\aleph_\alpha$ provided it satisfies Axiom 1$_{(\aleph_\alpha)}$, and a distributive space provided it satisfies Axiom 2 of the author's paper *Spaces in which there exist uncountable convergent sequences of points*. C. W. Vickery, op. cit., pp. 12–13.
radius $1/n$ having restrictions at $h$. Let $F$ denote the family of all collections $G_{(r,n)}$. It can easily be shown that $F$ is the required family. Axiom 2 is evidently satisfied.

Note. If in space $T^a$ the point $P$ be said to be the sequential limit point of a type $\omega_a$ sequence of points $[P_i]^{\omega_a}$ if and only if it is true that if $R$ is a region containing $P$, then $R$ contains a residue of sequence $[P_i]^{\omega_a}$, there then exist type $\omega_a$ convergent sequences of points. Thus we have been led by a series of apparently natural definitions to the existence of uncountable convergent sequences of points in certain spaces of uncountably many dimensions.

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NOTE ON THE LOCATION OF ZEROS OF THE DERIVATIVE OF A RATIONAL FUNCTION WHOSE ZEROS AND POLES ARE SYMMETRIC IN A CIRCLE*

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1. Introduction. The most general function which effects a 1-to-$m$ conformal transformation of the interior of the unit circle $|z| = 1$ onto itself is of the form

$$r(z) = \lambda \prod_{k=1}^{m} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad |\alpha_k| < 1, \quad |\lambda| = 1;$$

so the location of the zeros of the derivative $r'(z)$ is of considerable interest. The zeros and poles of $r(z)$ are symmetric in the unit circle. Moreover a typical transcendental function bounded in the unit circle is the Blaschke product (assumed convergent)

$$B(z) = \prod_{k=1}^{\infty} \frac{\bar{\alpha}_k}{|\alpha_k|} \frac{z - \alpha_k}{\bar{\alpha}_k z - 1},$$

which is the limit for $|z| < 1$ of a sequence of functions each of form (1). It is of some significance in studying the behavior of $B(z)$ to know exactly or approximately the zeros of $B'(z)$. The object of the present note is to give some fairly simple but elegant results on the derivatives of both $r(z)$ and $B(z)$. Application is made also to the critical points of certain harmonic functions.

2. Derivative of a rational function. We first obtain the following result:

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