ON THE VALUES ASSUMED BY POLYNOMIALS

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The equation

\[
\frac{f(x + h) - f(x)}{h} = k
\]

(1)

where \( f(x) \) is an integral function and \( k \) is an independent complex variable, defines \( h \) as a many valued function of \( x \) and \( k \). The branches of this function \( h(x, k) \) possess a simple property when \( f(x) \) is a polynomial; and conversely, when the branches of \( h(x, k) \) have the property, \( f(x) \) must be a polynomial. It is the object of this paper to exhibit the property, prove that it is characteristic for polynomials, and extend the results to polynomials of several variables.

1. Polynomials in one variable. For a polynomial

\[
f(x) = a_0 x^{n+1} + a_1 x^n + \cdots + a_{n+1}, \quad a_0 \neq 0, \: n > 1,
\]
equation (1) becomes

\[
h^n + [(n + 1)x + a_1/a_0]h^{n-1} + \cdots - k/a_0 = 0,
\]
and the \( n \) branches \( h_i(x, k) \) of the function \( h(x, k) \) satisfy the equation

\[
\sum_{i=1}^{n} h_i(x, k) = -(n + 1)x - a_1/a_0.
\]

For \( n = 1 \) we readily find that \( h = -2x - a_1/a_0 + k/a_0 \). Hence we have

\[
\frac{\partial}{\partial x} \sum_{i=1}^{n} h_i = -(n + 1), \quad n \geq 1,
\]
as a property of polynomials of degree \( n+1 \). For \( n = 0 \) no function \( h \) is defined by (1).

THEOREM I. Let \( f(x) \) be an integral function. If there exist \( n \) functions \( h_1(x, k), \cdots, h_n(x, k) \) analytic, distinct, and nonzero on a region in the \( x \) plane and a region in the \( k \) plane, and which on these regions are such that

\[
\frac{f(x + h_i(x, k)) - f(x)}{h_i(x, k)} = k, \quad i = 1, \cdots, n,
\]
and

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\[
\frac{\partial}{\partial x} \sum_{1}^{n} h_i(x, k) = -(n + 1),
\]

then \( f(x) \) is a polynomial of degree \( n+1 \).

The \( n \) functions \( h_i \) of our hypothesis satisfy

\[
(h - h_1)(h - h_2) \cdots (h - h_n) = 0
\]

\[
= h^n - P_1 h^{n-1} + \cdots + (-1)^i P_i h^{n-i} + \cdots + (-1)^n P_n.
\]

In order to prove the theorem we establish relations between the elementary symmetric functions \( P_i \) and their partial derivatives. Every function \( h_i \) satisfies the equation

\[
\left(1 + \frac{\partial h}{\partial x}\right) = \frac{f'(x) - k \frac{\partial h}{\partial k}}{h}.
\]

We may derive this by differentiating both members of (1) with respect to \( x \), obtaining

\[
f'(x + h) \left(1 + \frac{\partial h}{\partial x}\right) = f'(x) + k \frac{\partial h}{\partial x}
\]

and with respect to \( k \) obtaining

\[
f'(x + h) \frac{\partial h}{\partial k} = h + k \frac{\partial h}{\partial k}.
\]

Since \( h \neq 0 \), \( \partial h / \partial k \neq 0 \), and we may solve this last equation for \( f'(x + h) \); and the substitution of the value thus obtained into the preceding equation yields (4). We shall make use of the fact that \( \partial h / \partial k \neq 0 \) later in the proof. From (4) we see that

\[
\sum_{1}^{n} \left(\frac{\partial h_i}{\partial k} \frac{1}{h_i}\right) = \frac{\partial P_n}{\partial k} \frac{1}{P_n} = \frac{1}{f'(x) - k} \sum_{1}^{n} \left(1 + \frac{\partial h_i}{\partial x}\right);
\]

and since by hypothesis \( \sum_{1}^{n} \partial h_i / \partial x = -(n+1) \), this reduces to

\[
\frac{1}{P_n} \frac{\partial P_n}{\partial k} = \frac{1}{k - f'(x)}.
\]

We now proceed to deduce a second equation connecting the \( P \)'s and their partial derivatives; namely,

\[
\frac{\partial P_{i+1}}{\partial x} = -(n-i+1)P_i + (k - f'(x)) \frac{\partial P_i}{\partial k},
\]

\[i = 1, 2, \ldots, n - 1.\]
To accomplish this observe that
\[ \frac{\partial P_{i+1}}{\partial x} + (n - i)P_i = S h_1 h_2 \cdots h_i \sum_i \left(1 + \frac{\partial h_i}{\partial x}\right) \]
where \( S \) denotes the sum of the terms arising by choosing the \( i \) indices of the \( h \)-factors in all possible ways among the numbers 1, 2, \( \ldots \), \( n \), and then letting \( j \) run through the remaining \( n - i \) numbers. The right-hand side of this equation can be written in the form
\[ S h_1 h_2 \cdots h_i \left\{ \sum_{j=1}^{n} \left(1 + \frac{\partial h_j}{\partial x}\right) - \sum_{j=1}^{i} \left(1 + \frac{\partial h_j}{\partial x}\right) \right\}. \]

With the aid of (2) and (4) the expression now becomes
\[ S h_1 h_2 \cdots h_i \left\{ \sum_{j=1}^{n} \left(1 + \frac{\partial h_j}{\partial x}\right) - \sum_{j=1}^{i} \left(1 + \frac{\partial h_j}{\partial x}\right) \right\} \]
\[ \frac{\partial P_{i+1}}{\partial x} + (n - i)P_i = -P_i + (k - f'(x)) \frac{\partial P_i}{\partial k}. \]

Hence II follows at once.

It will be convenient to change the notation, replacing \( i \) by \( n - i \). Then
\[ \frac{\partial P_{n-i+1}}{\partial x} = -(i + 1)P_{n-i} + (k - f'(x)) \frac{\partial P_{n-i}}{\partial k}, \]

II*
\[ i = 1, 2, \ldots, n - 1. \]

From I it follows that
\[ P_n = \phi(x)(k - f'(x)) \]
where \( \phi(x) \) is analytic in the given domain of the \( x \)-plane and does not vanish there. Observe that \( P_n \neq 0 \) since no \( h_i(x, k) \) vanishes, and, since the left-hand side of I is analytic in the given domain, \( k - f'(x) \) cannot vanish there.

We proceed to show that \( P_{n-i} \) is a polynomial in \( k \), at most of degree \( i+1 \), with coefficients which are analytic functions of \( x \) in the given domain, or else that \( P_{n-i} \) vanishes identically. We use the method of mathematical induction. First, we know that \( P_n \) is linear in \( k \). Next, observe that from II* it follows that
\[ \frac{\partial}{\partial k} \frac{P_{n-i}}{(k - f'(x))^{i+1}} = \frac{\partial P_{n-i+1}}{\partial x} \frac{1}{(k - f'(x))^{i+2}}. \]

Assume the statement true for \( P_n, P_{n-1}, \ldots, P_{n-i+1} \). Then the
right-hand side of this last equation can be represented by partial fractions
\[ \frac{A_{i+2}}{(k - f'(x))^{i+2}} + \cdots + \frac{A_2}{(k - f(x))^2}, \]
where the \( A_i \) are functions of \( x \) analytic in the given domain. The fraction whose denominator is \( k - f'(x) \) is absent, because the degree of the numerator, \( \partial P_{n-i+1}/\partial x \) is at most \( i \). Hence on integrating we have
\[ \frac{P_{n-i}}{(k - f'(x))^{i+1}} = -\frac{A_{i+2}}{(i + 1)(k - f'(x))^{i+1}} - \cdots - \frac{A_2}{k - f'(x)} + \psi(x). \]
This completes the proof.

If we substitute \((f(x+h)-f(x))/h\) for \( k \) in (3), and multiply both members by \( h \), we have an equation of the form
\[ h^{n+1} + \sum C_{pq}(x)h^p f(x+h)^q = 0, \quad p + q + r \leq n + 1. \]
This equation is of positive degree in \( f(x+h) \). For, if it were of degree zero, \( h \) would depend on \( x \) alone and \( \partial h/\partial k = 0 \). We have seen above that \( \partial h/\partial k \neq 0 \). Thus, for constant \( x \), \( f(x+h) \) coincides with a branch of an algebraic function of \( x+h \); and since \( f \) is integral, the algebraic function is a polynomial, and so is \( f \).

There remains only to show that \( f \) is a polynomial of degree \( n + 1 \). Since (1) defines \( n \) distinct, nonzero elements \( h_i \) for \( f \), the degree of \( f \) is not less than \( n + 1 \). In addition, we readily see the degree of \( f \) cannot exceed \( n + 1 \). In fact, if we substitute the Taylor expansion of \( f(x+h) \) in (5), the resulting polynomial in \( h \) must be identically zero, and this can happen only if the degree of \( f(x+h) \) in \( h \) does not exceed \( n + 1 \).

2. Polynomials in \( N \) variables. The above characterization of polynomials of a single variable finds its analogue for polynomials of several variables as follows: The equation (1) is replaced by
\[ f(x_1 + k_1 h, x_2 + k_2 h, \ldots, x_N + k_N h) - f(x_1, x_2, \ldots, x_N) = k \]
with \( x_1, \ldots, x_N, k_1, \ldots, k_N, k \) independent complex variables. If we denote by \( D \) the operator \( k_1 \partial/\partial x_1 + k_2 \partial/\partial x_2 + \cdots + k_N \partial/\partial x_N \), we may assert that if \( f(x_1, \ldots, x_N) \) is a polynomial in \( x_1, \ldots, x_N \), of total degree \( n + 1 \), then there exist \( n \) analytic, distinct, nonzero elements
\[ h_i(x_1, \ldots, x_N; k_1, \ldots, k_N; k) \]
which satisfy (1') and are such that
This fact is readily established if we expand \( f(x_1 + k_1 h, \ldots, x_N + k_N h) \) by Taylor’s theorem, obtaining
\[
f(x_1 + k_1 h, \ldots, x_N + k_N h) = f(x_1, \ldots, x_N) + hDf + \cdots + \frac{h^i}{i!} D^i f + \cdots + \frac{h^{n+1}}{(n+1)!} D^{n+1} f.
\]
Thus (1') may be written
\[
h^n + (n+1) \frac{D^nf}{D^{n+1}f} h^{n-1} + \cdots + \frac{(n+1)!}{i!} \frac{D^i f}{D^{n+1}f} h^{i-1} + \cdots \]
\[
\frac{k(n+1)!}{D^{n+1}f} = 0;
\]
and since \( D^{n+2}f = 0 \),
\[
D \sum_{i=1}^{n} h = D \left( - (n+1) \frac{D^nf}{D^{n+1}f} \right) = - (n+1) \left( \frac{D^{n+1}f}{D^{n+1}f} \right) = - (n+1).
\]

**Theorem II.** Let \( f(x_1, \ldots, x_N) \) be an integral function. If there exist \( n \) functions \( h_i(x_1, \ldots, x_N; k_1, \ldots, k_N; h) \), analytic, distinct, and non-zero on a region in \( (x_1, \ldots, x_N; k_1, \ldots, k_N; k) \) space and which on this set are such that they satisfy (1') and
\[
D \sum_{i=1}^{n} h = - (n+1),
\]
then \( f \) is a polynomial of total degree \( n+1 \).

The proof of Theorem II is identical with the proof of Theorem I, where the counterpart of equation (4) is
\[
(1 + Dh) = \frac{(Df - k)}{h} \frac{\partial h}{\partial k}
\]
and those of I and II are, respectively,
\[
I' \quad \frac{1}{P_n} \frac{\partial P_n}{\partial k} = - \frac{1}{Df - k},
\]
\[
\Pi' \quad DP_{i+1} = - (n - i + 1) P_i - \frac{\partial P}{\partial k} (Df - k).
\]
To establish (4') from (1') we operate on both members with \( D \), getting
\[ Df + kDh = Df(x_1 + k_1h, \ldots, x_N + k_Nh) \]

\[ = \sum k_i \frac{\partial}{\partial x_i} f(x_1 + k_1h, \ldots, x_N + k_Nh) \]

\[ = \left( \sum k_i \frac{\partial}{\partial x_i} \right) \cdot f(x_1 + k_1h, \ldots, x_N + k_Nh) \left( k_i \frac{\partial h}{\partial x_i} \right) \]

\[ + \sum k_i \frac{\partial}{\partial x_i + k_ih} f(x_i + k_ih) \]

\[ = \left( \sum k_i \frac{\partial}{\partial x_i + k_ih} \right) \cdot f(x_1 + k_1h, \ldots, x_N + k_Nh) \left( 1 + Dh \right). \]

Differentiating both members of (1') with respect to \( k \), we have

\[ k \frac{\partial}{\partial k} h + h = \left\{ \sum k_i \frac{\partial}{\partial x_i + k_ih} f(x_1 + k_1h, \ldots, x_N + k_Nh) \right\} \frac{\partial h}{\partial k}, \]

and eliminating the term in the bracket yields equation (4').

Since the operator \( D \) behaves for its variables like the operator \( d/dx \) for a single variable, all the reasoning of the previous section can, from this point on, be used verbatim, proving Theorem II.

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