CREMONA INVOLUTIONS DETERMINED BY A PENCIL OF SURFACES

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1. Introduction. The characteristics of the involutional Cremona transformations determined by a pencil of surfaces of order $n$ and containing an $(n - 2)$-fold line $d$ have been determined by Carroll [1]. The particular features of these transformations, which arise when the surfaces of the pencil are of order 3 and the curve residual to the line $d$ in the base of the pencil is composite, have been considered in some detail by the same author [2]. Snyder [3] has suggested that a similar study of involutions defined by surfaces of higher order might be of interest.

The transformation is defined by Carroll as follows. Let

$$\lambda_1 F_n(x) - \lambda_2 F_n'(x) = 0$$

be a pencil of surfaces of order $n$ containing the line $d \equiv x_1 = 0$, $x_2 = 0$ to multiplicity $n - 2$. Let $(z) = (0, 0, z_3, z_4)$ be a variable point on the line $d$, and let the pencil of surfaces (1) be connected with $(z)$ by the relation

$$z_0 \phi_1(\lambda_1, \lambda_2) - z_0 \phi_2(\lambda_1, \lambda_2) = 0,$$

where $\phi_i$, ($i = 1, 2$), is a binary form of order $k$. A point $(y)$ of space determines a surface of the pencil (1) and hence a value of the ratio $\lambda_1 : \lambda_2$, which in turn determines a point $(z)$ of $d$. The line joining $(y)$ and $(z)$ meets the member of (1) determined by $(y)$ in one further point $(y')$, the transform of $(y)$ in the involution. The characteristics of the transformation are

$$S_1 \sim S_{2n(k+1)-1} : d^{2(n-2)}(k+1)k(n - 2)d^2$$
$$C_{4n-4}^{2k+1} \{ (6n - 8)k + 6n - 10 \} g,$$

$$d \sim T_{2n(k+1)-2} : d^{2(n-2)}(k+1)k(n - 2)d$$
$$C_{4n-4}^{2k+1} \{ (6n - 8)k + 6n - 10 \} g,$$

$$C_{4n-4} \sim \Sigma_{4n(k+1)-4} : d^{4(n-2)}(k+1)k(n - 2)d^4$$
$$C_{4n-4}^{4k+1} \{ (6n - 8)k + 6n - 10 \} g^2,$$

$$R_{nk+2n-1} \sim R_{nk+2n-1} : d^{(n-2)}(k+2)k(n - 2)d$$
$$C_{4n-4}^{k+1} \{ (6n - 8)k + 6n - 10 \} g,$$

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where $C_{4n-4}$ is the base curve of the pencil of surfaces residual to $d$, $T$ and $\Sigma$ are the principal surfaces corresponding to $d$ and $C_{4n-4}$ respectively, $R$ is the surface of coincident points and the \{(6n-8)k+6n-10\} are fundamental curves of the second kind which are all lines.

2. Noether's map. Noether [4] has shown that a surface of order $n$ containing an $(n-2)$-fold line can be mapped on a plane by means of a web of curves of order $n$ and genus $n-2$ whose base is made up of a fixed $(n-2)$-fold point $N$ and $3n-4$ simple points $p_i$, ($i=1, \cdots, 3n-4$). That is,

$$[N^{n-2}p_1p_2 \cdots p_{3n-4}]^n \sim C_n,$$

the generic plane section of the surface $S_n$. Contained in the web of plane curves is the pencil $[N^1][N^{n-3}p_1p_2 \cdots p_{3n-4}]^{n-1}$. But the pencil of lines $[N^1]$ corresponds to the pencil of conics $C_2$ cut out on the surface by a pencil of planes through the $(n-2)$-fold line $d$. Hence the uniquely determined curve $[N^{n-3}p_1p_2 \cdots p_{3n-4}]^{n-1} \sim d$. Another member of the pencil of surfaces meets $S_n$ in a curve of order $n^2$ made of $d$ counted $(n-2)^2$ times and the residual curve $C_{4n-4}$ of order $4n-4$. The map of the complete intersection in the plane is of the form

$$[N^{n(n-2)}p_1^np_2^np_3^{n-1}]^{n^2}.$$  

Hence

$$C_{4n-4} \sim [N^{3n-6}p_1^2p_2^2p_3^{2n-4}]^{3n-2}.$$  

It will happen $3n-4$ times that a line of the pencil on $N$ coincides with one of the $3n-4$ lines $[Np_i]^1$. In this case the map of the plane section becomes $[Np_i]^1p_i^2[N^{n-4}p_1p_2 \cdots p_{3n-4}]^{n-1}$, which corresponds to the $(n-2)$-fold line $d$ and a composite conic made up of the $2$ lines $u_i \sim [Np_i]$ and $v_i \sim p_i^*$, the directions about the base point $p_i$. The $6n-8$ lines $u_i$, $v_i$, ($i=1, 2, 3, \cdots, 3n-4$), which lie by pairs in $3n-4$ planes through the $(n-2)$-fold line $d$, are the only lines on the surface $S_n$. Noether mentions that, aside from these lines and the pencil of conics $C_2$, to which they belong, there are no curves of order less than $n-2$ on the surface. The curve $C_{4n-4}$ is of genus

$$(3n-3)(3n-4)/2 - (3n-6)(3n-7)/2 - (3n-4) = 6n-11.$$  

It meets $d 4n-8$ times, each $u_i$ and $v_i$ 3 times and each $C_2$ 4 times.

It is the purpose of this paper to examine those cases which arise when $C_{4n-4}$ becomes composite and to determine what effect is pro-
duced on the order of the transformation and the configuration of its fundamental curves.

3. Lines as components of base. Let the map of $C_{4n-4}$ be of the form

$$p_1^* p_2^* p_3^* \cdots p_m^* [N^{3n-6} p_1 p_2 p_3 \cdots p_m p_{m+1} \cdots p_{3n-4}]^{3n-2}.$$ 

Then $C_{4n-4} = v_1 v_2 \cdots v_m C_{4n-m-4}$. $C_{4n-m-4}$ is of genus $6n-2m-11$, and meets $d$, $4n-m-8$ times and $v_i$, $(i = 1, 2, 3, \cdots, m)$, $3$ times. Hence it meets the plane $(dv_i)^1$ in one variable point. Through any point $(y)$ of this plane there can be drawn one and only one line meeting $d$, $v_i$ and $C_{4n-m-4}$ each once and hence lying entirely on the surface of the pencil $(1)$ determined by $(y)$. Now by $(2)$, to each point of $d$ there corresponds $k$ surfaces of the pencil $(1)$. From any point on $d$ there can be drawn one line meeting $v_i$ and $C_{4n-m-4}$. This determines a $(1, k)$ algebraic correspondence of valence zero which has $k+1$ coincidences. Therefore, it happens $k+1$ times that a line joining a point $(y)$ to the point $(z)$ of $d$ corresponding to the surface of the pencil $(1)$ determined by $(y)$ lies entirely on that surface. These lines are fundamental curves of the second kind of the type $(d v_i C_{4n-m-4})$. There are $3n-m-4$ lines $u_i$ and $3n-m-4$ lines $v_i$, $(i = m+1, m+2, \cdots, 3n-4)$, on each surface $S_n$ meeting $C_{4n-m-4}$ twice. From any point of $d$ there can be drawn

$$(4n-m-5)(4n-m-6)/2 - (4n-m-8)(4n-m-9)/2 - (6n-2m-11) = 6n-m-10$$

lines meeting $C_{4n-m-4}$ twice each. Hence there are $(6n-2m-8)k + 6n-m-10$ $F$-lines of the type $(d C_{4n-m-4})$. The remaining $mk$ $F$-lines of the transformation are accounted for by each $v_i$, $(i = 1, 2, 3, \cdots, m)$, counting $k$ times as an $F$-curve of the second kind, as well as an $F$-curve of the first kind. In order to determine the maximum possible number of lines $v_i$ which may break away from $C_{4n-4}$, we note that the residual curve can exist on the surface $S_n$ if and only if its map can exist in the plane under the prescribed conditions on its base. That is, $(3n-2)(3n+1)/2 - (3n-6)(3n-5)/2 - 6m-3(3n-m-4) \geq 0$ or $m \leq 2n-2$. The same results are obtained if the map of $C_{4n-4}$ is of the form

$$[N p_1] [N p_2] \cdots [N p_m] [N^{3n-m-6} p_1 \cdots p_m p_{m+1} \cdots p_{3n-4}]^{3n-m-2}$$

except that $v_i$ is replaced by $u_i$, $(i = 1, 2, 3, \cdots, m)$. 
4. Base having conics as components. Let the map of $C_{4n-4}$ be of the form

$$\left\{\left[\frac{N}{N}\right]\right\}^m \cdot \left[\frac{N_{3n-m-6}}{p_1 p_2 \cdots p_{2n-4}}\right]^{3n-m-2}.$$ 

It corresponds to $C'_1 C'_2 \cdots C_{2(m)} C_{4n-2m-4}^{(3n-m-11)}$, that is, to $m$ conics and a curve of order $4n - 2m - 4$ and genus $6n - 3m - 11$. Each conic meets $d$ twice, and $C_{4n-2m-4}$ meets $d$ $4n - 2m - 8$ times and each conic $4$ times. There are $(6n - 8)k + 6n - 3m - 10$ $F$-lines of the type $(dC_{4n-2m-4})$ and no others. Hence each such conic reduces the number of $F$-lines of the second kind by $3$. An examination of (3) shows that the plane of each such conic factors once out of each member of the homaloidal web and the $P$-surface $T_{2n(k+1)-3}$ of the transformation, and twice out of the $P$-surface $\Sigma_{4n(k+1)-4}$. Consequently, if there are $m$ such conics, the order of the transformation is reduced to $2n(k+1) - m - 1$. To determine the maximum possible number of such conics, we have

$$(3n - m - 2)(3n - m + 1)/2 - (3n - m - 6)(3n - m - 5)/2$$

$$- 3(3n - 4) = 6n - 5m - 4 \geq 0 \text{ or } m \leq (6n - 4)/5.$$ 

That is, $m = n + j$ when $n = 5j + k$, $(h = 4, 5, 6, 7, 8; j = 0, 1, 2, 3, \ldots)$. Obviously, $C_{4n-4}$ cannot be composed entirely of conics of this type for any value of $n$, since each conic must meet $d$ twice, while $C_{4n-4}$ meets $d$ but $4n - 8$ times. If we impose the condition that $2m - 4n - 8$, then $2n - 4 \leq (6n - 4)/5$ and $n \leq 4$. For $n = 3, 4$, $C_{4n-4}$ is composed of $2n - 4$ conics and a $C_{4,1}$ which meets each conic $4$ times but does not meet $d$. If $n = 3$ the order of the transformation is $6k + 3$ and there are $10k + 2$ $F$-lines; if $n = 4$ the order is $8k + 3$ and there are $16k + 2$ $F$-lines. The same results are obtained if some of the conics are composite, since the map of the curve residual to the conics is the same as before.

5. One nonplanar component in base. Let $C_{4n-4}$ be composed of a single nondegenerate curve of order $N_1 \geq n - 2$ and conics and lines of the type mentioned above. Its map is then of the form:

$$[N' p_1^2 p_2^2 \cdots p_s^2 p_{s+1}^r \cdots p_{s+t}^r] \left\{\left[\frac{N}{N}\right]\right\}^m \left[\frac{N_{1}}{p_{s+1}}\right]^1 \cdots \left[\frac{N_{p_{s+t}}}{1}\right]^1,$$

where

$$(4) \quad m + t + j = 3n - 2, \quad s + r + j = 3n - 6, \quad s + j = 3n - 4,$$

and

$$(5) \quad (m - 1)(m - 2)/2 - r(r - 1)/2 - s \geq 0,$$

$$m(m + 3)/2 - r(r + 1)/2 - 3s - j \geq 0.$$
From these conditions it follows that

\[ 4 \leq m \leq 3n - 2, \quad r = m - 4, \quad m - 2 \leq s \leq 3m - 9, \]

\[ 0 \leq j = 3n - s - 4 \leq 5m - 3s - 6, \quad 0 \leq t = s - m + 2. \]

Any set of integers which satisfies (6) will then constitute the characteristic of the map of a possible form of \( C_{4n-4} \). The genus of the curve \( C_N \), is \( p = 3m - s - 9 \), and

\[ N_1 = nm - (n - 2)(m - 4) - 2s - j = 4n + 2m - 2s - j - 8. \]

The curve \( C_{N_1} \) meets \( d(n - 1)m - (n - 3)(m - 4) - 2s - j = N_1 - 4 \) times. There are 2s of the lines \( u_i, v_i, (i = 1, 2, \ldots, s) \), which meet \( C_{N_1} \) twice each. From any point on \( d \) there can be drawn

\[ (N_1 - 1)(N_1 - 2)/2 - (N_1 - 4)(N_1 - 5)/2 - p \]

\[ = 12n + 3m - 5s - 3j - 24 \]

lines which meet \( C_N \) twice each. Hence, there are \( 2sk + 12n + 3m - 5s - 3j - 24 \), or \( 2sk + 3N_1 - 3m + s \) \( F \)-lines of the type \( (dC_N^2) \). The curve \( C_{N_1} \) meets each of the planes \( (du_i)^1 \), \( (i = s + 1, \ldots, s + j) \), in one variable point. Hence, there are \( j(k + 1) \) \( F \)-lines of the type \( (dC_N u_i) \). Each of the lines \( u_i \) counts \( k \) times as an \( F \)-curve of the second kind.

6. **Two nonplanar components in base.** Let \( C_{4n-4} \) be composed of two curves of orders \( N_1 \geq n - 2 \) and \( N_2 \geq n - 2 \) and of genera \( p_1 \) and \( p_2 \) respectively, besides conics and lines. Let the map of \( C_{N_1} \) have the characteristic \( (m_1r_1s_1j_1) \) where \( m_1 \) is the order, \( r_1 \) is the multiplicity at \( N_1 \), \( s_1 \) is the number of double points at base points \( p_i \), and \( j_1 \) is the number of simple points at base points \( p_i \). Let the map of \( C_{N_2} \), have the characteristic \( (m_2r_2s_2j_2) \) with respect to the same base, and let the mapping curves have \( a \) simple points in common. Then

\[ m_1 + m_2 + t + j_1 + j_2 - 2a = 3n - 2, \]

\[ r_1 + r_2 + t + j_1 + j_2 - 2a = 3n - 6, \]

\[ s_1 + s_2 + j_1 + j_2 - a = 3n - 4, \]

where \( t \) is the number of conics which meet \( d \) twice. From the first two of the relations (7) \( m_1 - r_1 + m_2 - r_2 = 4 \), whence either \( r_1 = m_1 - 1 \), \( r_2 = m_2 - 3 \), or \( r_1 = m_1 - 2 \), \( r_2 = m_2 - 2 \). In the first case

\[ r_1 = m_1 - 1, \quad r_2 = m_2 - 3, \]

\[ s_1 = 0, \quad 2m_2 - s_2 - 5 \geq 0, \]

\[ 2m_1 - j_1 \geq 0, \quad 4m_2 - 3s_2 - j_2 - 3 \geq 0. \]
Hence

\[ 1 \leq m_1 \leq 3n - 5, \quad 3 \leq m_2 \leq 3n - m_1 - 2, \]
\[ 0 \leq s_2 \leq 2m_2 - 5, \]
\[ 0 \leq j_1 \leq 2m_1, \quad 0 \leq j_2 \leq 4m_2 - 3s_2 - 3, \]
\[ 0 \leq a = j_1 + j_2 + s_2 - 3n + 4, \quad 0 \leq t = a + s_2 - m_1 - m_2 + 2. \]

(9)

Any set of integers satisfying the conditions (8) and (9) constitutes the characteristic of the map of a possible form of \( C_{4n-4} \). Then \( N_1 = n + 2m_1 - j_1 - 2, p_1 = 0, N_2 = 3n + 2m_2 - 2s_2 - j_2 - 6, \) and \( p_2 = 2m_2 - s_2 - 5. \) The curve \( C_{N_1} \) meets \( dN_1 - 1 \) times and \( C_{N_2} \) meets it \( N_2 - 3 \) times.

The \( F \)-curves of the second kind are arranged as follows:

\[
(s + a)k + 2N_2 - 2m_2 + s_2 \text{ of the type } (dC_{N_2}^s),
\]
\[
(s + a)k + 3N_1 + N_2 - 3m_1 - m_2 + a \text{ of the type } (dC_{N_1}C_{N_2}),
\]
\[
(j_1 - a)(k + 1) \text{ of the type } (duC_{N_1}),
\]
\[
(j_2 - a)(k + 1) \text{ of the type } (duC_{N_2}),
\]
\[
j_1 + j_2 - 2a \text{ lines each counted } k \text{ times.}
\]

In the second case:

\[ r_1 = m_1 - 2, \quad r_2 = m_2 - 2, \]
\[ m_1 - s_1 - 2 \geq 0, \quad m_2 - s_2 - 2 \geq 0, \]
\[ 3m_1 - 3s_1 - j_1 - 1 \geq 0, \quad 3m_2 - 3s_2 - j_2 - 1 \geq 0. \]

Hence

\[ 2 \leq m_1 \leq 3n - 4, \quad 2 \leq m_2 \leq 3n - m_1 - 2, \]
\[ 0 \leq s_1 \leq m_1 - 2, \quad 0 \leq s_2 \leq m_2 - 2, \]
\[ 0 \leq j_1 \leq 3m_1 - 3s_1 - 1, \quad 0 \leq j_2 \leq 3m_2 - 3s_2 - 1, \]
\[ 0 \leq a = s_1 + s_2 + j_1 + j_2 - 3n + 4, \quad 0 \leq t = a + s_1 + s_2 - m_1 - m_2 + 2. \]

Any set of integers satisfying (11) and (12) will then constitute the characteristic of the map of a possible form of \( C_{4n-4} \), except one containing the characteristic (3 1 0 8). These cases must be barred because this is the characteristic of the map of the multiple line \( d \) which cannot form a part of \( C_{4n-4} \). We have \( N_1 = 2n + 2m_1 - 2s_1 - j_1 - 4, \)
\[ p_1 = m_1 - s_1 - 2, \]
\[ N_2 = 2n + 2m_2 - 2s_2 - j_2 - 4, \]
\[ p_2 = m_2 - s_2 - 2. \]

The curves \( C_{N_1} \) and \( C_{N_2} \) meet \( dN_1 - 2 \) and \( N_2 - 2 \) times, respectively.

The \( F \)-curves of the second kind are arranged as follows:
\[ (s_1 + s_2)k + N - m_1 + s_1 \text{ of the type } (dC^2_{N_1}), \]
\[ (s_1 + s_2)k + N - m_2 + s_2 \text{ of the type } (dC^2_{N_2}), \]
\[ 2ak + 2N_1 - 2N_2 - 2m_1 + 2m_2 + a \text{ of the type } (dC_{N_1}N_2), \]
\[ (j_1 - a)(k + 1) \text{ of the type } (dC_{N_1}U_i), \]
\[ (j_2 - a)(k + 1) \text{ of the type } (dC_{N_2}U_i), \]
\[ (j_1 + j_2 - 2a)U_i \text{ each counted } k \text{ times}. \]

7. Three nonplanar components in base. In addition to possible conics and lines, let \( C_{4n-4} \) be composed of 3 curves \( C_{N_1P_1}, C_{N_2P_2}, C_{N_3P_3} \), whose maps have the characteristics \( (m_1r_1s_1t_1), (m_2r_2s_2t_2), \) and \( (m_3r_3s_3t_3) \), respectively. Let the maps of \( C_{N_1} \) and \( C_{N_2} \) have \( a \) simple base points in common, those of \( C_{N_2} \) and \( C_{N_3} \) \( b \) such points, and those of \( C_{N_1} \) and \( C_{N_3} \) \( c \) such points. Then

\[ m_1 + m_2 + m_3 + t + j_1 + j_2 + j_3 - 2a - 2b - 2c = 3n - 2, \]
\[ r_1 + r_2 + r_3 + t + j_1 + j_2 + j_3 - 2a - 2b - 2c = 3n - 6, \]
\[ s_1 + s_2 + s_3 + j_1 + j_2 + j_3 - a - b - c = 3n - 4. \]

From the first 2 of the relations (14) it follows that, without loss of generality, we can set \( r_1 = m_1 - 1, r_2 = m_2 - 1, r_3 = m_3 - 2. \) Then

\[ s_1 = 0, \quad s_2 = 0, \quad m_3 - s_3 - 2 \geq 0, \]
\[ 2m_1 - j_1 \geq 0, \quad 2m_2 - j_2 \geq 0, \quad 3m_3 - 3s_3 - j_0 - 1 \geq 0. \]

Hence

\[ 0 \leq m_1 \leq 3n - 5, \quad 0 \leq m_2 \leq 3n - m_1 - 4, \quad 0 \leq m_3 \leq 3n - m_1 - m_2 - 2, \]
\[ 0 \leq s_3 \leq m_3 - 2, \]
\[ 0 \leq j_1 \leq 2m_1, \quad 0 \leq j_2 \leq 2m_2, \quad 0 \leq j_3 \leq 3m_3 - 3s_3 - 1, \]
\[ 0 \leq a + c \leq j_1, \quad 0 \leq a + b \leq j_2, \quad 0 \leq b + c \leq j_3, \]
\[ 0 \leq a + b + c = j_1 + j_2 + j_3 + s_3 - 3n + 4, \]
\[ 0 \leq t = a + b + c + s_3 - m_1 - m_2 - m_3 + 2. \]

Any set of integers satisfying (15) and (16) will constitute the characteristic of the map of a possible form of \( C_{4n-4} \) except one containing the characteristic \((3 1 0 8)\). We have \( N_1 = n + 2m_1 - j_1 - 2, \) \( p_1 = 0, \)
\( N_2 = n + 2m_2 - j_2 - 2, \) \( p_2 = 0, \)
\( N_3 = n + 2m_3 - 2s_3 - j_3 - 4, \) and \( p_3 = m_3 - s_3 - 2. \) The curves \( C_{N_1}, C_{N_2}, \) and \( C_{N_3} \) meet \( d \) \( N_1 - 1, \) \( N_2 - 1, \) and \( N_3 - 2 \) times, respectively. The \( F \)-curves of the second kind are arranged thus:
(s_3 + a)k + N_1 + N_2 - m_1 - m_2 + a of the type \((dC_{N_1}C_{N_2})\),
(s_3 + a)k + N_3 - m_3 + s_3 of the type \((dC_{N_3})\),
(b + c)k + 2N_1 + N_3 - 2m_1 - m_3 + c of the type \((dC_{N_1}C_{N_3})\),
(b + c)k + 2N_2 + N_3 - 2m_2 - m_3 + b of the type \((dC_{N_2}C_{N_3})\),
\((j_1 - a - c)(k + 1) of the type \((dC_{N_1}u_i)\),
\((j_2 - a - b)(k + 1) of the type \((dC_{N_2}u_i)\),
\((j_3 - b - c)(k + 1) of the type \((dC_{N_3}u_i)\),
\((j_1 + j_2 + j_3 - 2a - 2b - 2c) u_i each counted k times.

8. Four nonplanar components in base. Finally, let \(C_{s_n-4}\) be made
up of 4 curves \(C_{N_h}p_h\) \((h = 1, 2, 3, 4)\), aside from conics and lines, and
let the map of \(C_{N_h}\) in the plane have the characteristic \((m_hr_hs_hj_h)\).
Let the maps of \(C_{N_1}\) and \(C_{N_3}\) have a simple base points in common,
those of \(C_{N_1}\) and \(C_{N_3}\) such points, those of \(C_{N_1}\) and \(C_{N_4}\) such points,
those of \(C_{N_2}\) and \(C_{N_3}\) such points, those of \(C_{N_2}\) and \(C_{N_4}\) such points,
and those of \(C_{N_3}\) and \(C_{N_4}\) such points.
Then
\[
\begin{align*}
m_1 + m_2 + m_3 + m_4 + t + j_1 + j_2 + j_3 + j_4 \\
- 2a - 2b - 2c - 2d - 2e - 2f = 3n - 2,
\end{align*}
\]
\[
\begin{align*}
r_1 + r_2 + r_3 + r_4 + t + j_1 + j_2 + j_3 + j_4 \\
- 2a - 2b - 2c - 2d - 2e - 2f = 3n - 6,
\end{align*}
\]
\[
\begin{align*}
s_1 + s_2 + s_3 + s_4 + j_1 + j_2 + j_3 + j_4 \\
- a - b - c - d - e - f = 3n - 4.
\end{align*}
\]
From these conditions it follows that
\[
\begin{align*}
r_h &= m_h - 1, \quad s_h = 0, \quad 2m_h - j_h \geq 0, \quad h = 1, 2, 3, 4.
\end{align*}
\]
Hence
\[
\begin{align*}
1 \leq m_1 &\leq 3n - 5, & 0 \leq j_h &\leq 2m_h, \\
1 \leq m_2 &\leq 3n - m_1 - 4, & 0 \leq a + b + c &\leq j_1, \\
1 \leq m_3 &\leq 3n - m_1 - m_2 - 3, & 0 \leq a + d + e &\leq j_2, \\
1 \leq m_4 &\leq 3n - m_1 - m_2 - m_3 - 2, & 0 \leq b + d + f &\leq j_3, \\
& & 0 \leq c + e + f &\leq j_4, \\
0 \leq t &= a + b + c + d + e + f - m_1 - m_2 - m_3 - m_4 + 2 \leq 3n - 6.
\end{align*}
\]
Any set of integers satisfying conditions (19) and (20) will constitute
the characteristic of a possible map of $C_{4n-4}$. We have $N_h = n + 2m_h - j_h - 2$ and $p_h = 0$, ($h = 1, 2, 3, 4$). The curve $C_{N_h}$ meets $d$ $N_h - 1$ times. The $F$-lines of the second kind may be classified as follows:

$$
\begin{align*}
(a+f)k + N_1 + N_2 - m_1 - m_2 + a & \text{ of the type } (dC_{N_1}C_{N_2}), \\
(b+e)k + N_1 + N_3 - m_1 - m_3 + b & \text{ of the type } (dC_{N_1}C_{N_3}), \\
(c+d)k + N_1 + N_4 - m_1 - m_4 + c & \text{ of the type } (dC_{N_1}C_{N_4}), \\
(c+d)k + N_2 + N_3 - m_2 - m_3 + d & \text{ of the type } (dC_{N_2}C_{N_3}), \\
(b+e)k + N_2 + N_4 - m_2 - m_4 + e & \text{ of the type } (dC_{N_2}C_{N_4}), \\
(a+f)k + N_3 + N_4 - m_3 - m_4 + f & \text{ of the type } (dC_{N_3}C_{N_4}), \\
(j_1 - a - b - c)(k+1) & \text{ of the type } (dC_{N_1}u_i), \\
(j_2 - a - d - e)(k+1) & \text{ of the type } (dC_{N_2}u_i), \\
(j_3 - b - d - f)(k+1) & \text{ of the type } (dC_{N_3}u_i), \\
(j_4 - c - e - f)(k+1) & \text{ of the type } (dC_{N_4}u_i), \\
(j_1 + j_2 + j_3 + j_4 - 2a - 2b - 2c - 2d - 2e - 2f)u_i & \text{ each counted } k \text{ times.}
\end{align*}
$$

9. Conclusion. All the possible forms which $C_{4n-4}$ may take have been determined. It has been shown that if $C_{4n-4}$ contains $m$ conics each meeting the multiple line $d$ twice, then the order of the transformation is reduced by $m$, but that no other form reduces that order. The configuration of the $F$-curves of the second kind varies widely from case to case, but their total number is always $(6n-8)k+6n-3m-10$.

Bibliography


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