GROUPS OF MOTIONS IN CONFORMALLY FLAT SPACES. II

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1. Introduction. In a previous paper with a similar title,* we have shown that all groups of motions admitted by a conformally flat metric space \( V_n \) must be subgroups of the general conformal group \( G_N \) of \( N = \frac{1}{2}(n+1)(n+2) \) parameters generated by

\[
(1) \quad \xi^i = b^i + a^i_0 x^i + x^i a^i_j x^j - \frac{1}{2} a^i_j \epsilon_i j(x^j)^2 + b^i x^i, \quad \epsilon_i = \pm 1.
\]

In (1), the \( b^i \) satisfy the relations \( e_i b^i + e_j b^j = 0, (i, j \text{ not summed}). Otherwise the \( a^i_0 \) and \( b^i \) in (1) are arbitrary.

To define a group of motions of \( V_n \), the \( \xi^i \) must satisfy the equations

\[
(2) \quad \xi^k \frac{\partial h}{\partial x^k} + h \frac{\partial \xi^i}{\partial x^i} = 0, \quad \text{\( i \) not summed},
\]

and the coordinates \( x^i \) of (2) are such that \( g_{ij} = \delta_{ij} h^2 \). Hence in this coordinate system, the metric has the form

\[
(3) \quad ds^2 = h^2 \sum e_i (dx^i)^2.
\]

In this paper we shall consider the simplest subgroups of \( G_N \), and determine the nature of the function \( h \) corresponding to each. Also we give a restatement of Theorem 2 of I, since it is not complete as given.

2. The group \( G_N \). The basis of the group \( G_N \) may be taken in the form

\[
(4) \quad P_i = \dot{p}_i,
\]

\[
(5) \quad S_{ij} = e_i x^i \dot{p}_j - e_j x^j \dot{p}_i, \quad i, j \text{ not summed},
\]

\[
(6) \quad U = x^i \dot{p}_i,
\]

\[
(7) \quad V_i = 2 x^i x^j \dot{p}_j - e_i e_j (x^j)^2 \dot{p}_i,
\]

where \( \dot{p}_i = \partial / \partial x^i \); and its commutators are ‡

*Groups of motions in conformally flat spaces, this Bulletin, vol. 42 (1936), pp. 418–422. The results of this paper (which we refer to as I) will be assumed known.
† All small Latin indices take the values 1, 2, \( \cdots \), \( n \), with \( n > 2 \), unless otherwise noted.
\( (8a) \quad (P_i, P_j) = 0, \)
\( (8b) \quad (P_i, U) = P_i, \)
\( (8c) \quad (P_i, S_{ik}) = e_\delta_{ij}P_k - e_\delta_{ik}P_i, \)
\( (8d) \quad (P_i, V_i) = 2\delta_{ij}U - 2e_\delta S_{ij}, \)
\( (8e) \quad (S_{ij}, S_{kl}) = e_\delta_{ik}S_{ij} - e_\delta_{ik}S_{ik} - e_\delta_{ik}S_{kl} + e_\delta_{ik}S_{jk}, \)
\( (8f) \quad (S_{ij}, U) = 0, \)
\( (8g) \quad (S_{ij}, V_k) = e_\delta_{ik}V_i - e_\delta_{ik}V_i, \)
\( (8h) \quad (U, V_i) = V_i, \)
\( (8i) \quad (V_i, V_j) = 0. \)

The four types of symbols, \( P_i, S_{ij}, U, V_i, \) will be considered singly and in various combinations to form the subgroups to be discussed.

3. **Subgroups of one type of symbol.** We consider first the subgroups with symbols

(a) \([P_a]\),  
(b) \([U]\),  
(c) \([S_{\alpha\delta}]\),  
(d) \([V_\alpha]\).

The notation \([P_a]\) means \([P_1, P_2, \cdots, P_r]\), and similarly for other expressions of this nature. That each of (a)–(d) forms a subgroup follows from (8a), (8e), (8i).

For (a), we have from (4), \( \xi^k = \delta^k_\alpha \), and (2), written in the form

\[ \frac{\partial h}{\partial x^k} + h \frac{\partial \xi^i}{\partial x^i} = 0, \]

becomes

\[ \frac{\partial h}{\partial x^\alpha} = 0. \]  

Hence (a): \( h = h(x^{r+1}, \cdots, x^n) \). In case \( r = n \), \( h \) is constant, and the \( V_n \) is flat.

The finite equations of the group \([P_a]\) are

\[ x'^i = x^i + a^\alpha \delta^i_\alpha \]

with parameters \( a^\alpha \). Because of the form of (10), we call this group the \( T_r \) of translations. However, the group of motions \([P_a]\) is not a group of translations of the \( V_n \) unless \( h = \) constant, \( \dagger \) that is, unless \( V_n \) is flat.

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* Greek letters take the values 1, 2, \cdots, \( r \), with \( r \leq n \).

† L. P. Eisenhart, *Continuous Groups of Transformations*, p. 212. We refer to this book as CG.
For (b), we have $\xi^i = x^i$, and (2) becomes

\begin{equation}
\frac{\partial h}{\partial x^i} = -h.
\end{equation}

Hence $h$ is homogeneous of degree $-1$, that is,

\begin{equation}
h = \frac{1}{x_1} \phi\left(\frac{x^2}{x_1}, \cdots, \frac{x^n}{x_1}\right),
\end{equation}

say, where $\phi$ is an arbitrary function of its arguments.

The finite equations of the group $[U]$ are $x'^i = ax^i$, the group of dilations.

For (c), we find

\begin{equation}
\xi^i_{\alpha \beta} = e_\alpha \delta^i_\beta x^\alpha - e_\beta \delta^i_\alpha x^\beta,
\end{equation}

as the vector components of the group $[S_{\alpha \beta}]$ of $\frac{1}{2}r(r-1)$ parameters. The equations (2) which must be satisfied for each $\xi^i_{\alpha \beta}$ now become

\begin{equation}
X_{\alpha \beta} h = e_\alpha x^\alpha \frac{\partial h}{\partial x^\beta} - e_\beta x^\beta \frac{\partial h}{\partial x^\alpha} = 0, \quad \alpha, \beta \text{ not summed},
\end{equation}

These equations have as general solution,

\begin{equation}
h = h(u; x^{r+1}, \cdots, x^n),
\end{equation}

where $u = \sum e_\alpha (x^\alpha)^2$.

In obtaining this, we use the fact that the system (12) contains $r - 1$ independent equations, since

\begin{equation}
e_\alpha x^\alpha X_{\beta \gamma} + e_\beta x^\beta X_{\gamma \alpha} + e_\gamma x^\gamma X_{\alpha \beta} = 0,
\end{equation}

and it is also a complete system.*

The group $[S_{\alpha \beta}]$ has the finite equations

\begin{equation}
x'^\alpha = a_\alpha^\beta x^\beta, \quad x'^A = x^A, \quad A = r + 1, \cdots, n,
\end{equation}

with

\begin{equation}
\sum e_\alpha a_\alpha^\beta a^\gamma_\beta = e_\rho \delta^\gamma_\rho.
\end{equation}

We call this group of $\frac{1}{2}r(r-1)$ parameters, the $R_{r(r-1)/2}$ of rotations.†

The vector components for the group (d) are

\begin{equation}
\xi^i_{\alpha} = 2x^i x^\alpha - e_\alpha \delta^i_\alpha R,
\end{equation}

† CG, p. 57, problem 12.
where $R = \sum e_i(x^i)^2$. Equations (2) reduce, for this case, to

\begin{equation}
2x^a x^i \frac{\partial h}{\partial x^i} - e_a R \frac{\partial h}{\partial x^a} + 2hx^a = 0.
\end{equation}

If we put $\dot{\lambda} = x^i \partial h / \partial x^i$, (13) may be written in the form

\[
\frac{2(\lambda + \ddot{h})}{R} = \frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha}, \quad \alpha \text{ not summed}.
\]

Since the left member of this equation is independent of $\alpha$, we may write

\[
\frac{e_\alpha}{x^\alpha} \frac{\partial h}{\partial x^\alpha} = \frac{e_\beta}{x^\beta} \frac{\partial h}{\partial x^\beta},
\]

which simplifies to (12), and hence $h$ is of the form for (c). Using this form for $h$ in (13), we obtain on reduction,

\begin{equation}
(u - v) \frac{\partial h}{\partial u} + \sum x^A \frac{\partial h}{\partial x^A} = -h, \quad A = r + 1, \ldots, n,
\end{equation}

with

\[
v = \sum e_A (x^A)^2.
\]

The equation (14) has as solution

\[
h = \frac{1}{R} \phi \left( \frac{x^{r+1}}{R}, \ldots, \frac{x^n}{R} \right).
\]

In case $r = n$, $h = a/R$, with $a$ constant, and the $V_n$ is flat.*

The finite equations for the group $[V_a]$ are†

\[
x^i = \frac{x^i - \frac{1}{2} R \delta_\alpha e_\alpha a_\alpha}{1 - a_\alpha x^\alpha + \frac{1}{4} e_\alpha e_\beta a_\alpha^2 (x^\beta)^2}.
\]

4. Subgroups with two types of symbols. We consider in this section the simplest subgroups with two types of symbols. These are:

(e) $[P_\alpha, S_{\beta\gamma}]$,  
(f) $[P_\alpha, U]$,  
(g) $[S_{\alpha\beta}, U]$,  
(h) $[V_a, U]$,  
(i) $[S_{\alpha\beta}, V_{\gamma}]$.  

Each of these we discuss briefly.

(e). The function $h$ has the same form as for (a) since equations (12) are satisfied identically if (9) are.

* L. P. Eisenhart, Riemannian Geometry, p. 85.
† Lie, loc. cit., p. 350.
(f). Using the form of \( h \) for (a) in (11), we see that \( h \) is homogeneous of degree \(-1\) in \( x^{r+1}, \ldots, x^n \), that is, we may write

\[
h = \frac{1}{x^{r+1}} \phi \left( \frac{x^{r+1}}{x^{r+1}}, \ldots, \frac{x^n}{x^{r+1}} \right).
\]

If \( r = n \), there is no solution.

(g). If we substitute for \( h \) in (11) its value as determined from (c), we obtain

\[
2u \frac{\partial h}{\partial u} + x^A \frac{\partial h}{\partial x^A} = -h, \quad A = r + 1, \ldots, n.
\]

Hence,

\[
h = \frac{1}{u^{1/2}} \phi \left( \frac{x^{r+1}}{u^{1/2}}, \ldots, \frac{x^n}{u^{1/2}} \right).
\]

(h). Equations (11) and (13) show \( \partial h/\partial x^a = 0 \), so that \( h \) is the same as in (f). If \( r = n \), there is no solution.

(i). For (d), we have seen that (13) imply (11), that is, the form of \( h \) for (i) is the same as that for (d).

5. **Subgroups with three and four types of symbols.** Of the four possibilities \([P_a, S_{\beta\gamma}, V_\delta], [P_a, S_{\beta\gamma}, U], [P_a, V_\beta, U], [S_{a\beta}, V_\gamma, U]\), only the second and fourth give subgroups:

(j) \([P_a, S_{\beta\gamma}, U]\),

(k) \([S_{a\beta}, V_\gamma, U]\).

For (j), the \( P_a, S_{\beta\gamma} \) imply \( h = h(x^{r+1}, \ldots, x^n) \), and then \( U \) shows \( h \) is the same form as in (f). There is no solution of \( r = n \).

The form of \( h \) for (k) will be the same for (h), as follows from (i), that is, \( h \) will have the same form as for (f). If \( r = n \), there is no solution.

The simplest four type symbol subgroup is

(l) \([P_a, V_\beta, S_{\gamma\delta}, U]\).

It is easily seen that the solution for \( h \) is the same as for (f), and there is no solution for \( r = n \).

6. **Indices in different ranges.** So far, we have considered only subgroups whose symbol indices all have the same range, \( 1, \ldots, r \). In this section we discuss cases (e), (i), (j), (k), and (l) with the indices for the various types of symbols in different ranges.

Case (m): \([P_i, S_{\beta\delta}]\). Let \( i \) range through \( 1, \ldots, r \), and \( j, k \) through any set of \( t \) indices, \( s_1, s_2, \ldots, s_t \), with \( s_1 < s_2 < \cdots < s_t \). Then either:
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\[ s_t \leq r, \quad (m_2) \quad s_1 \leq r, \quad s_t > r, \quad (m_3) \quad s_1 > r. \]

For case \((m_1)\), equations (9) imply (12) with \(\alpha, \beta\) in the range \(s_1, s_2, \ldots, s_t\). Hence \(h\) has the same form as in (a).

In the second case, \((m_2)\), there must be a common index in \((1, \ldots, r)\) and \((s_1, \ldots, s_t)\), say \(\beta\). Then, in (8c), choose \(i=j=\beta\), and \(k=s_t\). This gives

\[ (P_\beta, S_{\beta}s_t') = e_\beta P_{s_t'}, \quad s' = s_t, \]

which is not in the set \(\{P_a\}\). Hence, this case is impossible.

For case \((m_3)\), the two sets of indices have no index in common, and we must have \(t \geq 2\). Without loss of generality, we may take the set \(s_1, \ldots, s_t\) to be \(r+1, r+2, \ldots, r+t\). The form of \(h\) is easily seen to be

\[ h = h(v_t; x^{r+t+1}, \ldots, x^n), \quad v_t = \sum_{r+1}^{r+t} e_f(x^f)^2. \]

Case (n): \([S_{jk}, V_t]\). As in case (m), there are three possibilities, only the first and third being possible. If we let \(i\) take the range \(1, \ldots, r\), then if \(s_t \leq r, h\) has the same form as for (d). If \(s_t > r\), we may let \(j, k\) have the range \(r+1, \ldots, r+t\). Then \(h\) must satisfy (13), and (12) with the indices in this latter range. Since (13) implies (12), we must have \(h = h(u; v_t; x^{r+t+1}, \ldots, x^n)\). Using this form for \(h\) in (13), we obtain

\[ (u - w) \frac{\partial h}{\partial u} + v_t \frac{\partial h}{\partial v_t} + x^B \frac{\partial h}{\partial x^B} = -h, \quad B = r + t + 1, \ldots, n, \]

with \(w = \sum e_B(x^B)^2\). This equation has as solution

\[ h = \frac{1}{R - v_t} \phi \left( \frac{v_t}{R - v_t}; \frac{x^{r+t+1}}{R - v_t}, \ldots, \frac{x^n}{R - v_t} \right). \]

With three types of symbols, we consider first \([P_i, S_{jk}, U]\), and let \(i=1, \ldots, r\). If the indices of \(S_{jk}\) are all contained in the range \(1, \ldots, r\), \(h\) has the same form as for \([P_a, U]\). Otherwise, we must have all \(j, k\) indices outside the range \(1, \ldots, r\). Then we have: (o) \([P_a, S_{jk}, U]\), and \(h = h(v_t; x^B)\), using the notation of case (n). With this value of \(h\) in (11) we obtain equation (15) with \(u\) replaced by \(v_t\). Hence,

\[ h = \frac{1}{v_t^{1/2}} \phi \left( \frac{x^{r+t+1}}{v_t^{1/2}}, \ldots, \frac{x^n}{v_t^{1/2}} \right). \]

As the next case we consider \([V_a, S_{jk}, U]\). If the \(j, k\) indices are included in \(1, \ldots, r\), we get the same form for \(h\) as in \([V_a, U]\). If not
we must have \( j, k \) in the range \( J, K \), to give: (p) \([V_a, S_{JK}, U]\). The symbols \( V_a, U \) imply \( h = h(x^{r+1}, \ldots, x^n) \), and then the symbols \( S_{JK} \) imply \( h = h(v_t; x^B) \), the same as in (o).

The other two possibilities \([P_t, S_{jk}, V_t]\), \([P_t, V_j, U]\) are easily shown to be impossible, no matter in what ranges we choose the indices of the various symbols.

For four types we have \([P_a, S_{jk}, V_t, U]\). If \( j, k \) are in the \( J, K \) range, we have a contradiction from \((P_a, V_t)\), no matter what range \( l \) has. The only other choice is \( j, k \) included in the \( 1, \ldots, r \) range. Then, from \((P_a, V_t)\), we must have \( l \) in this range also. This gives

\[
(q) \quad [V_t, S_{a'b'}, V_{\gamma'}, U], \quad \alpha', \beta', \gamma' \text{ range included in } 1, \ldots, r,
\]
and \( h \) has the same form as for (f), as easily follows.

7. Summary. We give here a summary of the various forms for \( h \) corresponding to the subgroups considered.

(a) \([P_a]\), \( \quad h = h(x^{r+1}, \ldots, x^n); \)

(b) \([U]\), \( \quad h = \frac{1}{x^1} \phi \left( \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right); \)

(c) \([S_{a\beta}]\), \( \quad h = h(\mu; x^{r+1}, \ldots, x^n); \)

(d) \([V_a]\), \( \quad h = \frac{1}{R} \phi \left( \frac{x^{r+1}}{R}, \ldots, \frac{x^n}{R} \right); \)

(f) \([P_a, U]\), \( \quad h = \frac{1}{x^{r+1}} \phi \left( \frac{x^{r+2}}{x^{r+1}}, \ldots, \frac{x^n}{x^{r+1}} \right), r = n, \text{ no solution}; \)

(g) \([S_{a\beta}, U]\), \( \quad h = \frac{1}{u^{1/2}} \phi \left( \frac{x^{r+1}}{u^{1/2}}, \ldots, \frac{x^n}{u^{1/2}} \right); \)

(m8) \([P_a, S_{ij}]\), \( \quad h = h(\mu; x^B); \)

(n8) \([V_a, S_{ij}]\), \( \quad h = \frac{1}{R - v_t} \phi \left( \frac{v_t}{R - v_t}; \frac{x^B}{R - v_t} \right); \)

(o8) \([P_a, S_{ij}, U]\), \( \quad h = \frac{1}{v_t^{1/2}} \phi \left( \frac{x^B}{v_t^{1/2}} \right); \)

(e) \([P_a, S_{\beta\gamma}]\), and (m3) \([P_a, S_{\beta'\gamma'}]\), \( h \) as in (a);

(i) \([S_{a\beta}, V_{\gamma}]\), \( \quad h = \frac{1}{v_t^{1/2}} \phi \left( \frac{x^B}{v_t^{1/2}} \right); \)

(h) \([V_a, U]\), \( \quad \alpha \) \([P_a, S_{\beta'\gamma'}, U]\), \( \quad (k) \quad [S_{a\beta}, V_{\gamma'}, U], \)

(l) \([P_a, V_{\beta}, S_{\gamma'}] U\), \( \quad \alpha \) \([P_a, S_{\beta'\gamma'}] U\), \( \quad (o1) \quad [P_a, S_{\beta'\gamma'}, U], \)

(p1) \([V_a, S_{\beta'\gamma'}, U]\), \( \quad \alpha \) \([V_a, S_{\beta'\gamma'}, V_{\gamma'}, U]\),
all have \( h \) as in (f);

\[(p3) \quad [V_\alpha, S_{IJ}, U], \quad h \text{ as in } (o_3).\]

In the above summary we have used the following notation:

\[
R = \sum e_i (x^i)^2, \quad u = \sum e_\alpha (x^\alpha)^2, \quad v_t = \sum e_t (x^t)^2,
\]

\( i = 1, \ldots, n; \) Greek letters have the range 1, \ldots, \( r; \) \( i, J = r+1, \ldots, r+t; \) \( A = r+1, \ldots, n; \) primed Greek letters have a range contained within 1, \ldots, \( r; \) \( B = r+t+1, \ldots, n. \)

8. Restatement of Theorem 2 of I. In the proof of this theorem, the possibility \( a_0 = b^i = a_t = 0 \) was omitted. In this case, \( \xi^i \) has the form \( \xi^i = b^i x^i, \) and the function \( f(R) \) is arbitrary. The group for this case is evidently the rotation group \([S_{ij}]\) of \( \frac{1}{2} n(n-1) \) parameters. It is not difficult to show that the subgroups corresponding to the two cases mentioned in the theorem are \([ce_i P_i + V_i, S_{jk}]\) for \( f(R) = (\alpha R + \beta)^2 \) and \([S_{ij}, U]\) for \( f(R) = \alpha R. \) We may thus state the corrected theorem in the form:

**Theorem.** Every metric space with quadratic form \( \sum e_i (dx^i)^2/f(R) \) admits the rotation group \([S_{ij}]\) as a group of motions. The only metric spaces with this quadratic form which admit other groups of motions are spaces of constant curvature, and \( f \) has the form \( f(R) = (\alpha R + \beta)^2, \) and the group is \([ce_i P_i + V_i, S_{jk}]\), and spaces with \( f(R) = \alpha R, \) in which case the group is \([S_{ij}, U]\).

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