ON THE BASIS THEOREM FOR INFINITE SYSTEMS
OF DIFFERENTIAL POLYNOMIALS*

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Introduction. Let ℵ be a differential field of characteristic zero.†
We consider an infinite system Σ of differential polynomials in the
letters y₁, ⋯, yₙ, the coefficients of the differential polynomials be­
ing in ℵ.‡

A finite set Φ of forms in Σ is called a basis of Σ if, for every form G
in Σ, there is a positive integer p, dependent on G, such that Gᵖ is in
the differential ideal of Φ. If a single p will serve for every G in Σ,
then we shall call the basis strong.

It has been shown that every system has a basis.§ Raudenbush has
shown further,|| that there exist systems, not every basis of which is
strong. It is now natural to ask whether or not every system of forms
contains at least one strong basis.

We answer this question in the negative by showing that even a
perfect differential ideal of forms may have no strong basis. The per­
fected differential ideal with which we work is the one generated by the
form uv in the two unknowns u, v.

We employ several ideas used by Raudenbush in the second of his
above mentioned papers.

1. The assumption. Consider a form¶ G every term of which is di­
visible by some uᵢvᵢ.** Let Σ be the set of all such forms G. Then Σ
is a differential ideal, and is perfect. For, if a form has a term free of,
say, every uᵢ, then every power of the form will have such a term.

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† For the definition of differential field, and other terms, see H. W. Raudenbush,
Ideal theory and differential equations, Transactions of this Society, vol. 36 (1934),
p. 361–368.
‡ Throughout the rest of this paper we shall use, as is customary, the term form
for differential polynomial.
§ For differential fields of meromorphic functions this was essentially shown by
J. F. Ritt in his book Differential Equations from the Algebraic Standpoint, American
cially §§ 7, 77. Following the work of Ritt, Raudenbush treated the case of the
general differential field of characteristic zero by purely algebraic methods. See
Raudenbush, loc. cit.
|| On the analog for differential equations of the Hilbert-Netto theorem, this Bulletin,
¶ For ℵ we can use any differential field of characteristic zero.
** Subscripts denote derivatives.
It is easy to see that \( \Sigma \) is the perfect differential ideal generated by \( uv \).

If \( \Sigma \) has a strong basis, then it has one consisting purely of forms \( u_i v_j \).

Let

\[
(1) \quad u_i v_j, \quad i + j \leq s,
\]

be a strong basis for \( \Sigma \), and let \( p \) be the associated positive integer. We work toward a contradiction.

We denote by \( \alpha \) a positive integer to be fixed later.

Consider the set of all forms

\[
(2) \quad u_{i_1} v_{j_1} \cdots u_{i_p} v_{j_p}, \quad i_1 + j_1 + \cdots + i_p + j_p = \alpha.
\]

Every such form has an expression \( \sum r_c c_s^p (\sum h=1 \alpha_{ph} u_{i_h} v_{i_h})^p \), where \( r \) is some positive integer, and the \( c_s \) and the \( \alpha_{ph} \) are rational numbers.*

Therefore, by our assumption on the nature of the basis (1) and the integer \( p \), every form (2) is in the differential ideal generated by the forms (1).

Hence each form (2) is a linear combination, with coefficients in \( F \), of forms

\[
(3) \quad \sum_{s=1}^r c_s (\sum_{h=1}^p \alpha_{ph} u_{i_h} v_{i_h})^p,
\]

\[
i + j \leq s, \quad i + j + k + i_1 + j_1 + \cdots + i_{p-1} + j_{p-1} = \alpha.
\]

Since the forms (2) are all linearly independent over \( F \), it follows that the number of distinct forms (2) cannot exceed the number of distinct forms (3).

We denote the number of distinct forms (2) by \( R_{p, \alpha} \), and the number of distinct forms (3) by \( Q_{p, \alpha} \). We thus have \( R_{p, \alpha} \leq Q_{p, \alpha} \).

In the next section we force the contradiction that \( R_{p, \alpha} > Q_{p, \alpha} \) for \( \alpha \) sufficiently large.

2. **The contradiction.** We consider those expressions (2) for which \( i_1 + j_1 = \nu \), \( 0 \leq \nu \leq \alpha \). The coefficient of \( u_i v_j \) in (2) is then

\[
(4) \quad u_{i_2} v_{j_2} \cdots u_{i_p} v_{j_p}, \quad i_2 + j_2 + \cdots + i_p + j_p = \alpha - \nu.
\]

The number of distinct forms (4) is \( R_{p-1, \alpha-\nu} \), and therefore the number of distinct symbols\(^\dagger\) (4) is not less than \( R_{p-1, \alpha-\nu} \). Since the number of expressions \( u_i v_j \) with \( i_1 + j_1 = \nu \) is \( \nu + 1 \), the total number of sym-

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* We can solve the equations \( (w_1 + \lambda w_2)^t = \sum_{i=0}^t C_{\lambda i} \lambda^i w_1^{t-i} w_2^i \), \( \lambda = 1, \cdots, t+1 \), for \( w_1 w_2^t \), obtaining \( w_1 w_2^t = \sum_{\lambda=1}^{t+1} d_{\lambda} (w_1 + \lambda w_2)^t \). Using this special case, we can show by induction that \( w_1 \cdots w_p = \sum_{\lambda=1}^{t+1} d_{\lambda} (\sum_{h=1}^p \lambda_{ph} w_h)^p \). Setting \( w_i = u_i v_i \), we obtain the desired representation of the forms (2).

\( ^\dagger \) Two distinct symbols (4) may represent the same form.
bol's (2) is not less than $\sum_{\rho=0}^{\alpha} (\nu+1) R_{\rho-1,\alpha-\rho}$. But not more than $(\rho!)^2$ distinct symbols (2) can represent the same form. Hence

$$R_{\rho,\alpha} \geq (\rho!)^{-2} \sum_{\rho=0}^{\alpha} (\nu+1) R_{\rho-1,\alpha-\rho}. \quad (5)$$

We now show that there exist positive numbers $b_{\rho}$, ($\rho = 1, 2, \cdots$), independent of $\alpha$, such that

$$R_{\rho,\alpha} \geq b_{\rho} (\alpha + 1)^{2\rho - 1}. \quad (6)$$

Obviously $R_{1,\alpha} = \alpha + 1$, so that (6) holds for $\rho = 1$. Suppose (6) holds for $\rho = m - 1$. Then, by (5), using $[x]$ to denote the greatest integer not exceeding $x$, we have

$$R_{m,\alpha} \geq (m!)^{-2} \sum_{\rho=0}^{\alpha} (\nu + 1) b_{m-1} (\alpha - \nu + 1)^{2m-3} \geq (m!)^{-2} b_{m-1} \sum_{\rho=0}^{\lfloor 3\alpha/4 \rfloor} (\nu + 1) (\alpha - \nu + 1)^{2m-3} \geq (m!)^{-2} b_{m-1} \sum_{\rho=\lfloor \alpha/4 \rfloor}^{\lfloor 3\alpha/4 \rfloor} ([\alpha/4] + 1) ([\alpha/4] + 1)^{2m-3} \geq (m!)^{-2} b_{m-1} (2^{\lfloor \alpha/4 \rfloor} + 1) ([\alpha/4] + 1) (\alpha/4 + 1)^{2m-3} \geq b_m (\alpha + 1)^{2m-1},$$

where $b_m = (m!)^{-2} 4^{-2m} b_{m-1}$. Thus (6) holds for all $\rho$.

We now consider those expressions (3) for which $i + j + k = \mu$, ($0 \leq \mu \leq \alpha$). The number of distinct expressions $(u,v_j)_k$ with $i + j + k = \mu$ and with $i + j \leq s$ does not exceed $(s + 1)^2$. The coefficient of $(u,v_j)_k$ in (3) is $u_i v_{j_1} \cdots u_{i_{p-1}} v_{j_{p-1}}$, $i_j + j + \cdots + i_{p-1} + j_{p-1} = \alpha - \mu$. Since the number of distinct forms of this kind is $R_{p-1,\alpha-\mu}$, we have for the total number of distinct forms (3):

$$Q_{\rho,\alpha} \leq \sum_{\mu=0}^{\alpha} (s + 1)^2 R_{p-1,\alpha-\mu}. \quad (7)$$

We shall show that, for $\rho = 1, 2, \cdots$,

$$R_{\rho,\alpha} \leq (\alpha + 1)^{2\rho - 1}. \quad (8)$$

For since $R_{1,\alpha} = \alpha + 1$, (8) holds for $\rho = 1$. Suppose (8) holds for $\rho = m - 1$. Looking at (2), it is easy to see that

$$R_{m,\alpha} \leq \sum_{\rho=0}^{\alpha} (\nu + 1) R_{m-1,\alpha-\rho}. \quad (9)$$
Therefore

\[ R_{m, \alpha} \leq \sum_{\nu=0}^{\alpha} (\nu + 1)(\alpha - \nu + 1)^{2m-3} \]

\[ \leq \sum_{\nu=0}^{\alpha} (\alpha + 1)(\alpha + 1)^{2m-3} = (\alpha + 1)^{2m-1}. \]

Thus (8) holds for all \( \alpha \).

Using (8) in (7), we find

\[ Q_{p, \alpha} \leq (s + 1)^2 \sum_{\mu=0}^{\alpha} (\alpha - \mu + 1)^{2p-3}, \]

so that

\[ Q_{p, \alpha} \leq (s + 1)^2 (\alpha + 1)^{2p-2}. \]

Comparing this with (6), we see that, for \( \alpha \) sufficiently large, \( R_{p, \alpha} > Q_{p, \alpha} \).