ON A CERTAIN CLASS OF SYMMETRIC HYPERSURFACES

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In the literature of algebraic geometry, relatively little is written on hypersurfaces \( H \) of order \( n \) and of \( r - 1 \) dimensions in \( S_r \), invariant under the symmetric permutation group \( G \) on \( r + 2 \) homogeneous coordinates whose sum is zero. Surfaces in \( S_3 \) invariant under the symmetric \( G_{120} \) have been studied by Emch [1]; the Clebsch diagonal surface has been discussed by Clebsch [2], Eckardt [3], and Ciani [4, 5, 6]. The Segre cubic variety in \( S_4 \) has been investigated by Segre [8]; Ciani [7] has developed properties of loci in \( S_4 \) invariant under the \( G_{720} \).

It is well known that the equation of such a hypersurface \( H \) is expressible uniquely in terms of the elementary symmetric functions \( p_i \) (of order \( i \) in \( x_1, \ldots, x_{r+2} \)), or in terms of the \( \Sigma \) functions \( \Sigma_n = \sum_{i=1}^{r+2} x_i^n \). With the condition \( \Sigma_1 = 0 \), any \( H \) is a member of the linear system

\[
\sum_{i=1}^{N} C_i \Sigma_2^a \Sigma_3^b \cdots \Sigma_{r+2}^d = 0
\]

where \( N \) is the number of nonnegative solutions of the Diophantine equation

\[
2a + 3b + \cdots + (r + 2)d = 0.
\]

It follows immediately that there is a unique hyperquadric \( H_2, \Sigma_2 = 0 \) or \( p_2 = 0 \), a unique cubic \( H_3, \Sigma_3 = 0 \) or \( p_3 = 0 \), a pencil of quartics \( H_4, \lambda \Sigma_4 + \mu \Sigma_5 = 0 \), a pencil of quintics \( H_5, \lambda \Sigma_5 + \mu \Sigma_6 = 0 \), and so on, in \( S_r \) invariant under \( G \).

Since there are no real points on \( \Sigma_n = 0 \) if \( n \) is even, values of \( n \) will be restricted to odd positive integers throughout the remainder of this paper.

Emch [1] has shown that the equation of any surface of odd order in \( S_3 \), invariant under the symmetric \( G_{120} \) on 5 \( x \)'s whose sum is zero necessarily has the form \( A \Sigma_3 + B \Sigma_5 = 0 \), which is equivalent to \( \lambda \Sigma_3 + \mu \Sigma_5 = 0 \). The obvious generalization of this statement is that any \( H \) of order \( n \) necessarily is a member of the linear system

\[
A \Sigma_3 + B \Sigma_5 + \cdots + D \Sigma_{r+2} = 0;
\]

if \( r = 2k + 1 \), and in case \( r = 2k \), of the system

\[
A \Sigma_3 + B \Sigma_5 + \cdots + C \Sigma_{r+1} = 0.
\]

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If \( r = 2k+1 \), all \( H \) of (3) contain the base \( \Sigma_2 = \Sigma_3 = \cdots = \Sigma_{r+2} = 0 \). This base is of order \( \alpha = (r+2) \cdot r!/2^k \cdot k! \) and of dimension \( k \), consisting of the \( \alpha \) subspaces into which the \( S_k \) \( [x_1 + x_2 = 0, x_3 + x_4 = 0, \cdots, x_1 + x_{r+1} = 0, x_{r+2} = 0] \) is carried by the substitutions of \( G \). Each \( S_k \) is invariant under a subgroup of \( G \) of order \( 2^{k+1} \cdot (k+1)! \).

If \( r = 2k \), all \( H \) of (4) contain the base \( \Sigma_2 = \Sigma_3 = \cdots = \Sigma_{r+1} = 0 \), of order \( \beta = (r+1)!/2^k \cdot k! \) and of dimension \( k \), consisting of the \( \beta \) subspaces into which the substitutions of \( G \) carry the \( S_k \) \( [x_1 + x_2 = 0, x_3 + x_4 = 0, \cdots, x_{r+1} + x_{r+2} = 0] \); each \( S_k \) is invariant under a subgroup of \( G \) of order \( 2^{k+1} \cdot (k+1)! \).

The hypersurfaces \( \Sigma_n = 0 \) have as double points only those points whose coordinates are proportional to \((n-1)\)th roots of unity, since at a double point the \( r+1 \) partial derivatives \( \partial \Sigma_n/\partial x_i = nx_i^{n-1} - nx_i^{n+1} \), \( i = 1, 2, \cdots, r+1 \), must vanish (with the dependence of \( x_{r+2} \) expressed by \( \Sigma_1 = 0 \)). It follows immediately that \( \Sigma_n = 0 \) has \( C_{r+1,k} \) real double points if \( r = 2k \), and has no real double points if \( r = 2k+1 \). Imaginary double points of \( \Sigma_n = 0 \) will occur whenever there is a set of \( r+2 \) \((n-1)\)th roots of unity, not all real, whose sum is zero.

If \( r = 2k+1 \), then none of the \( \alpha \) subspaces \( S_k \) passes through a double point of \( \Sigma_n = 0 \), since at a double point no \( x_i = 0 \).

If \( r = 2k \), then in each of the \( \beta \) subspaces \( S_k \) there are \((n-1)^k \) double points of \( \Sigma_n = 0 \), of which exactly \( 2^k \) are real; through each of the real double points pass \((k+1)! \) subspaces \( S_k \). If \( r = 2k \), and \( 2k+1 \) is a prime, then the \( \beta \) subspaces \( S_k \) divide into \((r-1)! \) sets of \((r+1)\)-tuples, each \((r+1)\)-tuple being transformable into another \((r+1)\)-tuple by a cyclic substitution of \( G \) of period \( r+1 \). (This is a generalization of the 6 quintuples of planes on the Segre cubic variety.)

The hypersurfaces \( \Sigma_n = 0 \) can contain no points of multiplicity greater than two, since not all the partial derivatives of higher order vanish at any point.

Eckardt [3] has given an admirable synthetic and analytic discussion of the properties of an Eckardt point of a surface in \( S_r \). The analytic generalization is immediate. Let a generalized Eckardt point \( E \) of a hypersurface \( F \), of order \( m \geq 2 \) and of \( r-1 \) dimensions in \( S_r \), be a simple point of \( F \) such that the hyperplane \( T \) tangent to \( F \) in \( E \) intersects \( F \) in a hypercone with a vertex at \( E \). We may so choose the polylateral of reference in \( S_r \) that \( E \) is \((1, 0, \cdots, 0) \), and \( T \) is \( x_2 = 0 \). Then the equation of \( F \) necessarily has the form

\[
(5) \quad x_2x_1^{n-1} + a_1x_2x_1^{n-2} + \cdots + a_{m-2}x_2x_1 + a_m = 0
\]

where \( a_i \) is a form of order \( i \) in \( x_2, x_3, \cdots, x_{r+1} \).

It follows immediately that the \( i \)th polar of \( E \) reduces into \( T \)
(x_2=0) and a hypersurface of order \(m-i-1\) which does not pass through \(E\).

Conversely, if the polar of a point \(P\) with respect to a hypersurface \(F\) of order \(m\) reduces into a hyperplane \(\pi\) passing through \(P\) and a hypersurface of order \(m-2\) not passing through \(P\), then if \(F\) does not reduce into \(\pi\) and a hypersurface of order \(m-1\), \(P\) is a generalized Eckardt point of \(F\) and \(\pi\) is tangent to \(F\) at the point \(P\).

On each hypersurface \(\Sigma_n=0\) in \(S_r\) there are \(C_{r+2,2}\) Eckardt points \(E_{ij}\) \((x_i=-x_j \neq 0; x_s=0, s \neq i, j)\), with the hyperplane \(x_i+x_j=0\) tangent to \(\Sigma_n=0\) at \(E_{ij}\). The polar of \(E_{ij}\) with respect to \(\Sigma_n=0\) is \(x_i^{n-1}-x_j^{n-1}=0\), which contains the hyperplanes \(x_i+x_j=0\) and \(x_i-x_j=0\). The latter is the axis of the perspectivity \((ij)\), with center at \(E_{ij}\), under which \(\Sigma_n=0\) and the polar of \(E_{ij}\) are invariant. The \((n-1)\)-fold locus \(x_i=x_j=0\) of the polar of \(E_{ij}\) contains \(C_{r,2}\) Eckardt points \(E_{st}\), \((s, t \neq i, j)\).

If \(r=2k\), the hyperplane \(x_i+x_j=0\) contains \(C_{r,k}\) real double points of \(\Sigma_n=0\), and \(x_i-x_j=0\) contains the remaining \(C_{r,k-1}\) real double points. Each real double point \(D\) of \(\Sigma_n=0\) is collinear with \((k+1)^2\) couples of points of \(\Sigma_n=0\), each couple consisting of an \(E_{ij}\) and the double point corresponding to \(D\) under \((ij)\). In each of the \(\beta\) subspaces \(S_k\) on \(\Sigma_n=0\) there are \(k+1\) points \(E_{ij}\), and through each \(E_{ij}\) there pass \(r!/2^k \cdot k!\) subspaces \(S_k\), while the three collinear points \(E_{ij}\), \(E_{ik}\), \(E_{jk}\) do not lie in a common \(S_k\) on \(\Sigma_n=0\). An \(S_k\) is the locus of points common to the \(k+1\) hyperplanes tangent to \(\Sigma_n\) at the \(k+1\) points \(E_{ij}\) in the \(S_k\).

If \(r=2k+1\), through each \(E_{ij}\) on \(\Sigma_n=0\) pass \((r+1)!/2^k \cdot k!\) subspaces \(S_k\), with \(k+1\) points \(E_{ij}\) in each \(S_k\).

The Eckardt point of a hypersurface \(F\) is in general of multiplicity \(r-1\) on the Hessian of \(F\); the Eckardt points of \(\Sigma_n=0\) are of multiplicity \((r-1)(n-2)\) on the Hessian of \(\Sigma_n=0\), whose equation is \(\sum_{i=1}^{r+2} x_i^{n-2}=0\).

**References**


CORRECTION TO "ON GREEN'S FUNCTIONS IN THE THEORY OF HEAT CONDUCTION IN SPHERICAL COORDINATES"

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In the article entitled On Green's functions in the theory of heat conduction by H. S. Carslaw and J. C. Jaeger (this Bulletin, vol. 45 (1939), pp. 407–413), a misprint is noted in the expression for $G$ on page 133 of my article On the operational determination of two dimensional Green's functions in the theory of heat conduction (this Bulletin, vol. 44 (1938), pp. 125–133), the correct expression for $G$ being

$$G = u + v = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \cos(n\theta - \theta_0) \int_{-\infty}^{\infty} \alpha e^{-ka^2t} \frac{H_n^{(1)}(\alpha r_0)}{U_n(\alpha a)} \cdot \left\{ J_n(\alpha r) U_n(\alpha a) - H_n^{(1)}(\alpha r) \left[ \alpha \frac{d}{dz} J_n(z) + hJ_n(z) \right]_{z=\alpha a} \right\} d\alpha,$$

where

$$U_n(\alpha a) = \left[ \alpha \frac{d}{dz} H_n^{(1)}(z) + hH_n^{(1)}(z) \right]_{z=\alpha a}.$$

When this correct expression is employed, formula (20), page 313, of the present paper becomes

$$G(r, \theta, \phi, t; r_0, \theta_0, \phi_0) = \frac{1}{8\pi(r r_0)^{1/2}} \sum_{n=0}^{\infty} (2n + 1) P_n(\cos \gamma)$$

$$\cdot \int_{-\infty}^{\infty} \alpha e^{-ka^2t} \frac{H_n^{(1)}(\alpha r_0)}{U_{n+1/2}(\alpha a)} \left\{ J_{n+1/2}(\alpha r) U_{n+1/2}(\alpha a) - H_n^{(1)}(\alpha r) \right\} \left[ \alpha \frac{d}{dz} J_{n+1/2}(z) + (h - 1/(2a))J_{n+1/2}(z) \right]_{z=\alpha a} d\alpha.$$