so that $\sum_{m=1}^{\infty} |A_m(f, 0)| = \infty$. It remains to show that $f(x) \in L$ which is easily seen since

$$\int_{-\pi}^{\pi} f(x) \, dx = \sum_{i=0}^{\infty} 2^{-i} \int_{-\pi}^{\pi} f_n(x) \, dx \leq \sum_{i=0}^{\infty} 2^{-i(2(n+1)} \frac{\pi}{3(n+1)} < \infty.$$ 

We notice that, since this function vanishes in the neighborhood of the origin, it coincides with a function having an absolutely summable Fourier series in the neighborhood of the origin, and therefore absolute summability $C(1)$ is not a local property.

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**COMPLETE REDUCIBILITY OF FORMS**

**RUFUS OLDENBURGER**

1. **Introduction.** We shall say that $F$ is a form in $r$ essential variables with respect to a field $K$ if $F$ cannot be brought by means of a non-singular linear transformation in the field $K$ to a form with less variables. Let $F$ be a form of degree $p$ written as $a_{i_1}\ldots x_{j_1} \ldots x_k$, $(i, j, \ldots, k=1, 2, \ldots, n)$. We arrange the coefficients of $F$ in a matrix $A$ whose $n^{p-1}$ columns are of the form

$$\begin{vmatrix}
a_{1j} & \cdots & \cdots & a_{1k} \\
a_{2j} & \cdots & \cdots & a_{2k} \\
\vdots & \ddots & \ddots & \vdots \\
a_{nj} & \cdots & \cdots & a_{nk}
\end{vmatrix}.$$ 

The index $i$ is associated with the rows of $A$ and the $p-1$ indices $j, \ldots, k$ are associated with the columns of $A$. We assume that the coefficients in $F$ are so chosen that $A$ is symmetric in the sense that the value of an element $a_{ij}\ldots k$ is unchanged under permutation of the subscripts. It can be shown that $F$ is a form in $r$ essential variables if and only if the rank of $A$ is $r$.

A form $F$ is said to be completely reducible in a field $K$ if $F$ splits

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in $K$ into a product of linear factors. Hočevar proved\(^3\) that a form $F$ with no repeated factors is completely reducible in the complex field if and only if $F$ divides each third order minor of its Hessian. It is obvious that this result of Hočevar is not valid for each field of numbers. A form $F$ of degree $p$ is said to be nonsingular with respect to $K$ if $F$ can be written as a linear combination of $p$th powers of linearly independent linear forms with coefficients in $K$. Elsewhere the author proved\(^4\) that the Hessian of a cubic form nonsingular with respect to $K$ factors in $K$ into linearly independent factors. For a field $K$ with characteristic different from 2, 3, and element $a \neq 0$, the product $ax_1x_2 \cdots x_n$ in $n$ independent variables $x_1, x_2, \cdots, x_n$ is the Hessian of the nonsingular cubic $C(a)$ where $6C(a) = ax_1^3 + x_2^3 + \cdots + x_n^3$.

We let $L_i = b_{ij}y_j$, $(i, j = 1, 2, \cdots, n)$, denote an arbitrary set of $n$ linear forms linearly independent with respect to $K$. We write $\Delta$ for the determinant of the matrix $(b_{ij})$. Applying the nonsingular linear transformation $x_1 = L_1, x_2 = L_2, \cdots, x_n = L_n$ to $C(1/\Delta^2)$ we obtain a form whose Hessian is $L_1L_2 \cdots L_n$. Hence each product of linearly independent linear forms is the Hessian of a nonsingular cubic form. We have proved the theorem which follows.

**Theorem 1.** Let $K$ be a field with characteristic not 2 or 3. A form $F$ of degree $n$ in $n$ essential variables is completely reducible in $K$ if and only if $F$ can be written as the Hessian of a cubic form nonsingular with respect to $K$.

If $F$ of Theorem 1 is completely reducible and $F$ is the Hessian of a nonsingular cubic form $C$, then $C = a_1L_1^3, (i = 1, 2, \cdots, n)$, and the linear forms $L_1, \cdots, L_n$ are the factors of $F$.

The utility of Theorem 1 is limited by the fact that the problem of representability of a form as the Hessian of a nonsingular cubic is unsolved. In the present paper we prove that a certain integer, called “minimal number,” associated with a completely reducible form $F$ of degree $n$ is not greater than $2^{n-1}$. From this property we obtain a solution of the problem of complete reducibility of cubic forms for a field $K$ with characteristic not 2 or 3.

2. **Minimal numbers and representations.** Elsewhere\(^5\) the author proved that each symmetric form $F$ of degree $p$ can be written for a

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field $K$ of order $p$ or more as a linear combination of $p$th powers of linear forms. Such a linear combination with $p$ terms we call a $p$-representation of $F$ with respect to $K$. A representation of $F$ with respect to $K$ with a minimum number of terms is called a minimal representation of $F$ with respect to $K$. The number of terms in such a representation we term the minimal number of $F$ with respect to $K$, and denote this number by $m(F)$.

**Theorem 2.** Let $K$ be a field with characteristic greater than $n$, and let $F$ be a form of degree $n$ completely reducible in $K$. Then $m(F) \leq 2^{n-1}$.

We write $p = 2^{n-1}$. Let $L_1, L_2, \ldots, L_p$ denote the different possible forms of the type $(x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n)$. Let $k_i = +1$ if $L_i$ contains an even number of minus coefficients, and $k_i = -1$ if $L_i$ contains an odd number of such coefficients. We consider the sum

\[
\frac{1}{2^{n-1}} \left[ \sum_{i=1}^{p} k_i L_i^n \right].
\]

Simple computation reveals that (1) is symmetric in the $x$'s. We consider a product $\Pi = \pm x_1^a \cdots x_r^d$ of degree $n$ with $r < n$ arising from the expansion of a term $k_i L_i^p$ in (1). Corresponding to the linear form $L_i$ there is a unique form $L_j, (j \neq i)$, in (1) obtainable from $L_i$ by changing the sign of $x_n$ in $L_i$. Then $k_j = -k_i$. The product $P = x_1^a \cdots x_r^d$ arising from $k_i L_i^p$ has a coefficient the negative of that in $\Pi$. Thus the terms involving the product $P$, where these terms arise from $k_i L_i^p$ and $k_j L_j^p$, vanish. It follows that the coefficient of $P$ in (1) is zero. It is obvious from the choice of the $k_i$ that the coefficient of $x_1 \cdots x_n$ in (1) is $n!$, whence (1) is a $p$-representation of $n! x_1 \cdots x_n$. Since a completely reducible form $F$ in $n$ essential variables is equivalent to this product under nonsingular linear transformations in $K$, and the minimal number is an invariant of $F$, we have $m(F) \leq 2^{n-1}$. It follows that if $F = L_1 L_2 \cdots L_n$ where $L_1, L_2, \ldots, L_n$ are linearly dependent linear forms, $m(F) \leq 2^{n-1}$.

3. **Complete reducibility of cubic forms.** In the present section we assume that the underlying field $K$ is such that when two forms are equal to each other for all values of the variables in $K$, corresponding coefficients of these forms are equal. In the case of cubic forms this means that the characteristic of $K$ is different from 2, 3. Evidently, a completely reducible cubic form is a form in not more than 3 essential variables. Since the minimal number of a binary cubic is not greater

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* Restricting the characteristic of $K$ to be greater than $n$ is equivalent to assuming that the characteristic of $K$ does not divide $n!$. 

than 3, the theory of complete reducibility of binary forms may read-
ily be supplied by the reader. In what follows we therefore consider
cubic forms in 3 essential variables only.

**Theorem 3.** A cubic form $F$ in 3 essential variables is completely re-
ducible with respect to a field $K$ if and only if

(a) The minimal number of $F$ with respect to $K$ is 4.

(b) If $\mu_i R_i^3$ is a minimal representation of $F$ with respect to $K$, then
roots $\sigma_i = (\mu_i/\mu_1)^{1/3}$ are in $K$ for each $i$, and for some choice of the roots
$\sigma_i$ we have $\sum_{i=1}^4 \sigma_i R_i = 0$.

A completely reducible cubic form $F$ in 3 essential variables is
equivalent under nonsingular linear transformations in the given field
to $T = xyz$. By Theorem 2, $m(T) \leq 4$. If $m(T)$ were 3, the form $T$
would be equivalent to $C = au^3 + bv^3 + cw^3$ in the variables $u, v, w$,
whence $T$ is nonsingular. For $T$ to be nonsingular it is necessary and
sufficient that the Hessian $H$ of $T$ split into linearly independent
linear factors $L, M$, and $N$ and under reduction of $H$ to canonical
form $uvw$, $T$ transform covariantly to a reduced form $C$. Since the
Hessian of $T$ is already in canonical form and $T \neq ax^3 + by^3 + cz^3$,
we have $m(T) \neq 3$. The minimal number of a form cannot be less than
the number of essential variables in the form, whence $m(T) = 4$. Hence
$m(F) = 4$.

It is easy to prove that if $\sum_{i=1}^r \lambda_i (x + \alpha_i y)^n = 0$, where the $\lambda$'s are not
zero, and $r \leq n + 1$, the $\alpha$'s can be grouped into sets $S_1, S_2, \ldots, S_p$
each of order 2 at least, where the $\alpha$'s in each set are equal; and if we
let $\lambda_i$ correspond to $\alpha_i$, the sum of the $\lambda$'s corresponding to the $\alpha$'s
in $S_i$ vanishes for each $i$ in the range 1, 2, $\cdots$, $p$. From this it follows
rather immediately that if

$$6xyz = \sum_{i=1}^4 \lambda_i (x + \alpha_i y + \beta \delta)^3,$$

the right member of (2) is

$$\left(\frac{1}{4ab}\right) \left\{ (x + ay + bz)^3 - (x + ay - bz)^3 
- (x - ay + bz)^3 + (x - ay - bz)^3 \right\}.$$  

It is readily verified that the coefficients of $x, y, z$ in a representa-
tion $\lambda_i L_i^3$, $(i = 1, 2, 3, 4)$, of $6xyz$ are different from zero, whence any
representation of $6xyz$ can be written as the right member of (2).
Thus each representation of $6xyz$ is of the type (3), and (3) is a repre-

[Oldenburger, Rational equivalence of a form to a sum of $p$th powers, Trans-
actions of this Society, vol. 44 (1938), pp. 219–249.]
sentation of $6xyz$ for each choice of $a$, $b$ not zero. Since the representations of each form equivalent to $6xyz$ under nonsingular transformations can be obtained from $6xyz$ by substitutions $x = L$, $y = M$, $z = N$ where $L$, $M$, $N$ are linearly independent linear forms, a cubic form $F$ in 3 essential variables is completely reducible if and only if each 4-representation of $F$ is of the type

$$k\left\{(L + aM + bN)^3 - (L + aM - bN)^3 - (L - aM + bN)^3 + (L - aM - bN)^3\right\},$$

where $k$, $a$, $b \neq 0$, and $L$, $M$, $N$ are linearly independent.

Let a cubic form $F$ in three essential variable be given by a minimal representation $\sum_{i=1}^{4} \mu_{i}R_{i}$. If $F$ is completely reducible, the forms $\mu_{i}R_{i}$ ($i$ not summed; $i = 1, 2, 3, 4$) are identically equal to the forms $\pm k[L \pm aM \pm bN]^3$ in some order and for some choice of $k$, $a$, $b$, $L$, $M$, and $N$. Then there exists an element $c$ in the given field $K$ such that $\rho_{i} = (c\mu_{i})^{1/3}$ are in $K$, and an ordering of the values of $i$ so that

$$L + aM + bN \equiv \rho_{1}R_{1}, \quad L + aM - bN \equiv - \rho_{2}R_{2},$$
$$L - aM + bN \equiv - \rho_{3}R_{3}, \quad L - aM - bN \equiv \rho_{4}R_{4}.$$

Equations (5) are solvable for $L$, $M$, $N$ if and only if $\sum_{i=1}^{4} \rho_{i}R_{i} \equiv 0$. Evidently there exists an element $c$ in $K$ so that roots $\rho_{i}$ in $K$ exist if and only if there exist roots $\sigma_{i} = (\mu_{i}/\mu_{c})^{1/3}$ in $K$. Theorem 3 is now proved.

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