TYPICALLY-REAL FUNCTIONS WITH
\[ a_n = 0 \text{ for } n \equiv 0 \pmod{4} \]

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1. Introduction. Let

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

be typically-real for \(|z| < 1\); that is, \(f(z)\) within this circle is regular and takes on real values when and only when \(z\) is real. In particular, if \(f(z)\) is univalent for \(|z| < 1\) and has real coefficients, it is also typically-real. We suppose in addition that

\[ a_n = 0 \quad \text{for } n \equiv 0 \pmod{4}. \]

In this paper we obtain sharp inequalities for the coefficients \(a_n\).

Sharp inequalities for \(a_n\) are already well known\(^2\) with the more restrictive condition

\[ a_n = 0 \quad \text{for } n \equiv 0 \pmod{2} \]

holding. In this case \(|a_n| \leq n\) with equality occurring for the odd function \((z + z^3)(1 - z^2)^{-1}\). If besides, \(f(z)\) is univalent and real on the real axis, the coefficients are bounded and satisfy\(^3\) the inequalities

\[ |a_{2m-1}| + |a_{2m+1}| \leq 2, \quad |a_3| \leq 1. \]

With the less restrictive condition (1.2) replacing (1.3) the author obtains the following new and sharp inequalities:

\[ |a_n| + 2^{-3/2}[(n - 2) |a_{2m}| + n |a_2|] \leq n, \quad m, n \text{ odd}, n > 1; \]

\[ |a_n| + 2^{-1/2}(n - 1) |a_2| \leq n, \quad n \text{ odd}; \]

\[ |a_n| + |a_2| \leq 2^{3/2}, \quad |a_2| \leq 2^{1/2}, \quad n \text{ even}. \]

In each case the equality sign holds for the typically-real function

\[ z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2} \sum_{n=1}^{\infty} \sin n\pi/4 \cdot z^n. \]

Since this function is also univalent for \(|z| < 1\), the inequalities above

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are sharp also for the class of univalent functions with real coefficients for which (1.2) holds.

Since (1.5) may be written in the form

\[ |a_{2m}| + |a_2| \leq 2^{3/2} \left( 1 - \limsup_{n \to \infty} \frac{|a_n|}{n} \right), \]

(1.7) will follow at once as well as the following theorem.

**Theorem.** If within the unit circle the typically-real function

\[ f(z) = z + \sum_{n=0}^{\infty} a_n z^n, \quad a_n = 0 \text{ for } n \equiv 0 \pmod{4}, \]

has \( \limsup_{n \to \infty} \frac{|a_n|}{n} = 1 \), then \( f(z) \) is an odd function; that is to say, \( a_n = 0 \) for \( n \equiv 0 \pmod{2} \).

In a recent paper the author discussed a similar problem when \( a_n = 0 \) for \( n \equiv 0 \pmod{p} \), \( p \) odd, and particularly for \( p = 3 \). The method used in that paper does not generalize completely to \( p > 3 \). Certain modifications in the method were necessary to take care of asymmetric phases which appear when \( p > 3 \), and these are given here for \( p = 4 \). The method appears to fail completely for \( p > 4 \).

2. **Proof of the inequalities.** Let \( 5f(re^{i\theta}) = v(r, \theta) \), for \( r < 1 \). Since \( f(z) \) is typically-real for \( |z| = r < 1 \),

\[ v(r, \theta) > 0 \text{ for } 0 < \theta < \pi, \quad v(r, \theta) < 0 \text{ for } \pi < \theta < 2\pi, \]

\[ v(r, \pi - \theta) = -v(r, \pi + \theta), \quad v(r, \theta) = -v(r, -\theta). \]

In what follows we shall write \( v(r, \theta) \) as simply \( v(\theta) \). Since also

\[ a_n = 0 \text{ for } n \equiv 0 \pmod{4}, \]

it follows that

\[ f(z) + f(ze^{\pi i/2}) + f(ze^{\pi i}) + f(ze^{3\pi i/2}) = 0, \]

and in particular the imaginary part of the left-hand member is zero. We write this as

\[ v(\theta) + v(\pi/2 + \theta) - v(\pi - \theta) - v(\pi/2 - \theta) = 0. \]

The coefficients of \( f(z) \) are given by

\[ a_n = \frac{2}{\pi r^n} \int_0^\pi v(\theta) \sin n\theta d\theta. \]

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Let
\[ \int_0^\pi v(\theta) \sin n\theta \, d\theta = \int_0^{\pi/4} + \int_{\pi/4}^{\pi/2} + \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} \]
\[ = I_1 + I_2 + I_3 + I_4. \]

In \( I_2 \) let \( \theta = \pi/2 - \phi \) and obtain
\[ I_2 = \int_0^{\pi/4} v(\pi/2 - \phi) \sin n(\pi/2 - \phi) \, d\phi. \]

In \( I_3 \) let \( \theta = \pi/2 + \phi \) and obtain
\[ I_3 = \int_0^{\pi/4} v(\pi/2 + \phi) \sin n(\pi/2 + \phi) \, d\phi. \]

In \( I_4 \) let \( \theta = \pi - \phi \) and obtain
\[ I_4 = \int_0^{\pi/4} v(\pi - \phi) \sin n(\pi - \phi) \, d\phi. \]

In \( I_1 \) substitute for \( v(\theta) \) the value obtained from (2.4). Combining the new forms for \( I_1, I_2, I_3, \) and \( I_4 \) we have
\[ \int_0^\pi v(\phi) \sin n\phi \, d\phi \]
\[ = \int_0^{\pi/4} \{A v(\pi - \phi) + B v(\pi/2 - \phi) + C v(\pi/2 + \phi)\} \, d\phi, \]
where for brevity we write
\[ A = \sin n(\pi - \phi) + \sin n\phi = 2 \sin n\pi/2 \cos n(\pi/2 - \phi), \]
\[ B = \sin n(\pi/2 - \phi) + \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 - \phi), \]
\[ C = \sin n(\pi/2 + \phi) - \sin n\phi = 2 \sin n\pi/4 \cos n(\pi/4 + \phi). \]

Thus
\[ \int_0^\pi v(\phi) \sin n\phi \, d\phi = 2 \sin n\pi/2 \int_0^{\pi/4} v(\pi - \phi) \cos n(\pi/2 - \phi) \, d\phi \]
\[ + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 - \phi) \cos n(\pi/4 - \phi) \, d\phi \]
\[ + 2 \sin n\pi/4 \int_0^{\pi/4} v(\pi/2 + \phi) \cos n(\pi/4 + \phi) \, d\phi \]
\[ = K_1 + K_2 + K_3. \]
In $K_1$ let $\phi = \pi/2 - \alpha$, in $K_2$ let $\phi = \pi/4 - \alpha$, and in $K_3$ let $\phi = \alpha - \pi/4$. Then
\[
\int_0^\pi \varphi(\phi) \sin n\phi d\phi = 2 \sin n\pi/2 \int_{\pi/4}^{\pi/2} \varphi(\pi/2 + \phi) \cos n\phi d\phi
\]
(2.13)
\[
\quad + 2 \sin n\pi/4 \int_0^{\pi/2} \varphi(\pi/4 + \alpha) \cos n\phi d\phi.
\]
Hence the formula (2.5) for the coefficients $a_n$ may be replaced by
\[
a_n = \frac{4}{\pi r^n} \left[ \sin n\pi/2 \int_{\pi/4}^{\pi/2} \varphi(\pi/2 + \phi) \cos n\phi d\phi 
\quad + \sin n\pi/4 \int_0^{\pi/2} \varphi(\pi/4 + \phi) \cos n\phi d\phi \right].
\]
(2.14)
In particular, since $a_1 = 1$ we have
\[
1 = \frac{4}{\pi r} \int_{\pi/4}^{\pi/2} \varphi(\pi/2 + \phi) \cos \phi d\phi 
\quad + \frac{2^{5/2}}{\pi r} \int_0^{\pi/2} \varphi(\pi/4 + \phi) \cos \phi d\phi.
\]
(2.15)
For even values of $n = 2k$, $k$ odd, we have
\[
a_{2k} = \frac{4(-1)^{k-1}}{\pi r^{2k}} \int_0^{\pi/2} \varphi(\pi/4 + \phi) \cos 2k\phi d\phi,
\]
whence follows the inequality (to be used later)
\[
\frac{4}{\pi} \int_0^{\pi/2} \varphi(\pi/4 + \phi) d\phi \geq r^{2m} |a_{2m}|,
\]
(2.17)
and in addition the equality
\[
\frac{4}{\pi} \int_0^{\pi/2} \varphi(\pi/4 + \phi) d\phi 
\quad = \frac{8}{\pi} \int_0^{\pi/2} \varphi(\phi + \pi/4) \cos^2 k\phi d\phi + (-1)^k r^{2k} a_{2k}.
\]
(2.18)
From (2.14) we have for odd values of $n$
\[ r^n \left| a_n \right| \leq \frac{4n}{\pi} \int_{\pi/4}^{\pi/2} v(\phi + \pi/2) \cos \phi d\phi + \frac{2^{5/2}}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) d\phi. \] (2.19)

With the aid of (2.18) the last inequality becomes

\[ r^n \left| a_n \right| + (-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \]
\[ \leq \frac{4n}{\pi} \int_{\pi/4}^{\pi/2} v(\phi + \pi/2) \cos \phi d\phi + \frac{2^{5/2}}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos^2 k\phi d\phi \]
\[ \leq \frac{4n}{\pi} \int_{\pi/4}^{\pi/2} v(\phi + \pi/2) \cos \phi d\phi + \frac{2^{5/2}k}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos \phi d\phi \]
\[ = (n-2k) \left[ \frac{4}{\pi} \int_{\pi/4}^{\pi/2} v(\phi + \pi/2) \cos \phi d\phi + \frac{2^{5/2}k}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos \phi d\phi \right] \]
\[ + 2k \left[ \frac{4}{\pi} \int_{\pi/4}^{\pi/2} v(\phi + \pi/2) \cos \phi d\phi + \frac{2^{5/2}k}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos \phi d\phi \right] \]
\[ - \frac{2^{5/2}(n-2k)}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos \phi d\phi, \]

whence, on account of the equalities (2.15), (2.18) with \( k = 1 \), and (2.17) for values of \( 2k < n \), we have

\[ r^n \left| a_n \right| + (-1)^{k-1}2^{-1/2}r^{2k}a_{2k} \]
\[ \leq r^{n} - \frac{2^{5/2}(n-2k)}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) \cos^2 \phi d\phi \] (2.20)
\[ = r^{n} - \frac{2^{1/2}}{4} (n-2k) \left[ \frac{4}{\pi} \int_0^{\pi/2} v(\phi + \pi/4) d\phi + r^2a_2 \right] \]
\[ \leq r^{n} - (2^{1/2}/4)(n-2k) [r^{2m} \left| a_{2m} \right| + r^2a_2]. \]

By considering the function \(-f(-z)\), which is also typically-real, we obtain an inequality similar to this last one except that \( a_2 \) and \( a_{2k} \) have been replaced by \(-a_2\) and \(-a_{2k}\). Consequently, on combining both inequalities and letting \( r \) approach one we have for \( k \) and \( n \) odd

\[ \left| a_n \right| + 2^{-3/2} \left[ (n-2k) \left| a_{2m} \right| \right] + \left| (n-2k)a_2 + (-1)^{k-1}2a_{2k} \right| \leq n, \quad 2k < n. \] (2.21)

In particular, for \( k = 1 \) we derive for \( n \) odd
(2.22) \[ |a_n| + 2^{-3/2}[(n-2)|a_{2m}| + n|a_2|] \leq n, \quad n > 1. \]

If in addition \( m = 1 \), then for \( n \) odd

(2.23) \[ |a_n| + 2^{-1/2}(n-1)|a_2| \leq n. \]

From (2.22) on dividing by \( n \) and letting \( n \to \infty \) we have

\[
|a_{2m}| + |a_2| \leq 2^{3/2} \left[ 1 - \limsup_{n \to \infty} \frac{|a_n|}{n} \right] \leq 2^{3/2},
\]

(2.24) \[ |a_3| \leq 2^{1/2}, \quad \limsup_{n \to \infty} \left| \frac{a_n}{n} \right| \leq 1 - 2^{-1/2} |a_2|. \]

Though (2.22), (2.23), and (2.24) hold for \( m \) either even or odd, the interesting inequalities are for \( n \) and \( m \) both odd. In this case they are sharp, as is seen from an inspection of the coefficients of the univalent function

\[ z(1 - 2^{1/2}z + z^2)^{-1} = 2^{1/2} \sum_{n=1}^{\infty} \sin n\pi/4 \cdot z^n. \]

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