A FINITELY-CONTAINING CONNECTED SET

P. M. SWINGLE

In a previous paper an example has been given of a set which, for every integer \( n \geq 2 \), is the sum of \( n \) mutually exclusive connected subsets, but which is not the sum of \( \text{infinitely many such subsets} \). Here it is proposed to give an example of a connected set which, for every integer \( n \geq 2 \), is the sum of \( n \) mutually exclusive \textit{biconnected} subsets but which is not the sum of \( \text{infinitely many such subsets} \), being thus a \textit{finitely-containing connected set}. The method used will be a modification of that used by E. W. Miller to obtain a biconnected set without a dispersion point. The \textit{hypothesis of the continuum is assumed}, and use is made of the axiom of Zermelo.

The method used by Miller is dependent primarily upon showing

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3 A connected set is defined here so as to contain at least two points. The example there given consists of a connected set which is the sum of \( \text{infinitely many mutually exclusive biconnected subsets} \), each with a dispersion point, and a limit point of these subsets which none of them contains.
4 Loc. cit., p. 395, Problem 7. This example also solves the questions raised in Problems 4, 5, and 6, pp. 394–395. Problem 2 was answered in part in American Journal of Mathematics, vol. 54 (1932), pp. 532–535. On p. 533 it is proved for \( n = 2 \) that \( E_n \) is the sum of \( m \) mutually exclusive biconnected subsets where \( m \) is an integer greater than \( n \). And it is said that the proof is similar for \( n > 2 \). For \( E_3 \) the proof depends upon constructing \( 3 \) biconnected sets, having only the origin in common. That a similar construction holds for any \( E_n \) \((n > 1)\), is seen as follows. The half cones \( x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = ax_n^2 \), \((n \geq 0, -\infty < a < \infty)\), of \( E_n \) are each \( n - 1 \)-dimensional surfaces. As each one is composed of concentric spheres \( x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = r^2 \) as is also \( E_{n-1} \), each half cone and \( E_{n-1} \) are topologically equivalent. As for \( n = 3 \), \( E_{n-1} \) is the sum of \( n \) biconnected sets, with only the origin in common, a mathematical induction proof will show that this is true for \( n > 3 \). For let the \( a \)'s be divided into \( C_{n+1,n} \) \((C_{n+1,n} \text{ is a binomial coefficient})\) mutually exclusive sets \( N_i, \ldots, N_c \) each dense in their sum. Let, for each \( a \) of \( N_i \) \((i = 1, \ldots, c)\), \( x_1^2 + x_2^2 + \cdots + x_{n-1}^2 = ax_n^2 \) be the sum of parts of the same \( n \) biconnected sets, where there is a total of \( n + 1 \) such sets \( B_i \), mutually exclusive except that they have the origin in common. Those \( B_i \)'s determined by \( N_i \) will be represented by the subscripts of that combination of \( 1, 2, \ldots, n + 1 \), taken \( n \) at a time, that \( i \) of \( N_i \) represents. Then the above is seen to be true.

178
the existence of a widely connected subset $M$ of an indecomposable continuum $K$. It is only the part of this subset $M$ which is contained within a square $Q_0$ which causes $M$ to be biconnected and it is this fact which enables us to show the existence of the desired set of this paper. We will take a countable infinity of mutually exclusive such squares plus interiors, $Q_1, Q_2, Q_3, \ldots$, each containing points of $K$ and having the relation with $K$ that Miller's square $ABCD$ has. We will use $Q_i$ as Miller does to show that a subset $B_{ni}$, $(i = 1, 2, \ldots, n + 1; n = 1, 2, 3, \ldots)$, of a set $M$ is biconnected. And $Q$ will be used to show that there cannot be infinitely many mutually exclusive such subsets of $M$.

Let $V$ be a countable subset of $K$, which is dense in $K \cdot (Q_1 + Q_2 + Q_3 + \cdots + Q)$. Let $V_{ij}$, $(i = 1, 2, 3, \ldots; j = 1, 2, \ldots, i + 1)$, be a countable subset of $V$ everywhere dense in $V$ and such that (a) $V_{ij} \cdot V_{ki}$ is dense in $V$ if $i \neq k$, (b) for any $i$ the $V_{i,k}$'s, $(k = 1, 2, \ldots, i + 1)$, are mutually exclusive, and (c) $V_{11} + V_{22} + \cdots + V_{i,i+1} = V$. For example $V_{11}$ and $V_{12}$ are mutually exclusive and $V_{11} + V_{12} = V$. Then $V_{11}$ is divided into three mutually exclusive subsets, each dense in $V$, one for each of the sets $V_{21}$, $V_{22}$, $V_{23}$ where $V_{2j}$ is composed of such a set plus a similar subset of $V_{12}$. Each one of these three mutually exclusive subsets of $V_{11}$ is then divided into four mutually exclusive sets, each dense in $V$, to obtain the parts of $V_{31}$, $V_{32}$, $V_{33}$, $V_{34}$ contributed by $V_{11}$.

Let a division of $V$ into infinitely many mutually exclusive subsets be $U_1, U_2, \ldots$, where each $U_t$, $(t = 1, 2, \ldots)$, is everywhere dense in $V$. Either (1) there exists a region $R$ of $Q$ and a $V_{ij}$ such that a $U_t$ contains $R \cdot V_{ij}$, or (2) there does not exist such an $R$. If (2) is true, $V_{ij} \cdot U_t \cdot V_{ii}$ is dense in $V \cdot Q$ for each $i, j, t$. Consider case (1). Suppose for example that $U_t$ contains $R \cdot V_{32}$. Let $R_1$ be any region contained in $R$. Then $U_1$ contains a subset of $V_{rj}$, $(r > 3)$, which is dense in $V_{rj} \cdot R_1$, since $V_{32} \cdot R_1$ contains such a subset because of (a) above. Hence $U_t$, $(t \neq 1)$, cannot contain a $V_{rj} \cdot R_1$, since $U_t$ and $U_1$ are mutually exclusive. Suppose now that there exist a $U_t$, $(t \neq 1)$, $U_2$ say, which contains a $V_{3j} \cdot R_1$, $(f \neq 2$, but equals 1 say), for some $R_1$ of $R$. Hence as above $U_t$, $(t \neq 2)$, does not contain a $V_{rj} \cdot R_3$, where $R_3$ is any region of $R_1$. There may exist now a $U_t$, $(t \neq 1, 2)$, $U_3$ say, which contains a $V_{3j} \cdot R_2$ for $f \neq 1$, 2 but $f = 3$ say. However since the $U_t$'s are contained in $V_{31} + V_{32} + V_{33} + V_{34}$, there cannot exist a region $R_3$ of $R_3$ and a $U_t$, $(t \neq 1, 2, 3)$, such that $U_t$ contains $R_3 \cdot V_{3j}$, $(f \neq 1, 2, 3)$, for $R_3 \cdot V_{34}$ must contain $R_3 \cdot (U_4 + U_5 + \cdots)$. Thus in this case there exists an $R_3$ of $R$ such that there are at most three $U_t$'s which contain a $V_{ij} \cdot R_3$, where $R_3$ is any region of $R_3$. Hence there exists an $R_3$ of $R$
and a $U_i, U''$ say, such that for every $V_{ij}$, $V_{ij} - V_{ij} \cdot U''$ is dense in $V \cdot R_a$. Therefore in both cases (1) and (2) above there exists a region $R''$ of $Q$ and a $U_i, U''$ say, such that for every $V_{ij}$, $V_{ij} - V_{ij} \cdot U''$ is dense in $V \cdot R''$.

The proof used by Miller to show that his widely connected set $M$ is biconnected is dependent upon having a countable subset $\Delta$ of $M$ and upon having a set of simple closed curves within the square $ABCD$ which have nothing in common with $M$ except points of $\Delta$. One of these simple closed curves is taken for each subset of $\Delta = V$ which is dense in $V \cdot R$, where $R$ is any region containing points of $V$. And the simple closed curves contain from the points of $V$ only points from this subset of $V \cdot R$. The set of such possible subsets is $c$, the power of the linear continuum.

Following the method of Miller arrange in a well ordered sequence the continua $C_a$ which separate $K$: $C_1, C_2, C_3, \ldots, C_a, \ldots$, $a < \Omega_a$, where $\Omega_a$ is the first transfinite ordinal number to correspond to the cardinal number $a$ of the linear continuum. Let the regions of $Q$ be well ordered as well as the possible divisions $D_1, D_2, \ldots, D_a, \ldots$ of $V$ into infinitely many mutually exclusive subsets $U_1, U_2, \ldots$. As the power of this set of regions and the power of the set of $D_a$'s are both $c$, let there be a one-to-one correspondence between each of these and the sequence $C_1, C_2, \ldots, C_a, \ldots$.

Choose for each $C_a$, having nothing in common with the interior of the square $Q$, a point set $M_{ia}$ for each $i$ and in each $Q$, construct a simple closed curve $J_{ia}$, exactly as Miller does for his $M$, using, for each $i$, $Q_i, V$ in place of his $(ABCD) \cdot \Delta$. Thus in $K$, exterior to $Q$, we have infinitely many mutually exclusive sets, $N_1, N_2, \ldots, N_a, \ldots$ say, each exactly similar to Miller's biconnected set $M$, except for $K \cdot Q$. In each region $R_a$ of $Q$ let a simple closed curve $J_a''$ be constructed, by a method similar to that used by Miller, so that each $V_{ij}$ is dense in $K \cdot J_a''$. Each infinite division $D_a$ above of $V$ determines a $U_a''$ and an $R_a''$ of $Q$ such that, for each $i, j$, $V_{ij} - V_{ij} \cdot U_a''$ is dense in $V \cdot R_a''$. In each $R_a''$ construct a simple closed curve $J_a''$ such that each $V_{ij}$ is dense in $K \cdot J_a''$ but $J_a'' \cdot U_a'' = 0$. For each $C_a$ separating $Q \cdot K$ choose for each $V_{ij}$ a point or vacuous set, according to whether or not $C_a \cdot V_{ij}$ is vacuous, obtaining for each such $C_a$ an $M_{ija}$ of $Q$ with the properties of Miller's $M_a$'s. No $J_{ia} + J_a''$ contains a point of an $M_{ija}$ and no two $M_{ija}$'s consist of the same point.

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6 E. W. Miller, loc. cit., p. 129.
The method used is dependent upon having chosen at any time during the process, under the hypothesis of the continuum, at most a countable infinity of points in $M \cdot (C_1 + C_2 + \cdots + C_5)$, where $M = N_1 + N_2 + \cdots + V + M_{111} + M_{121} + \cdots + M_{112} + M_{122} + \cdots$. This is true here just as it was for Miller's $M_a$'s. As the set of composants of $K$ is of the power of the linear continuum, new points can always be chosen for new $C_a$'s, and each choice can be made so that no composant contains more than one point of $M$.

The set $M$ is widely connected, for each $C_a$ contains at least one point of $M$ and no composant of $K$ contains more than one point of $M$. Let $B_{1g}$, $(g = 1, 2)$, contain all of $N_g + [V_{1g} + \sum_{a=1}^{10} M_{1ga}] \times Q$, and let in addition $B_{11}$ contain all the rest of $M$, with the exception of the rest of $M$ in $Q$, and let $B_{12}$ contain this. Hence $B_{11}$ and $B_{12}$ are mutually exclusive sets whose sum is $M$. Each is connected, for every $C_a$ contains a point of each. Just as Miller showed, each $B_{1g}$ is biconnected, for suppose that $B_{11}$, say, is the sum of the two mutually exclusive subsets $W_1$ and $W_2$. As $W_1 \cdot V$ must be dense in $Q_1 \cdot V$, there exists a $J_{1a} \cdot M$ of $Q_1$ contained entirely in $W_1 \cdot V$, according to the construction of the $J_{1a}$'s. As $B_{11}$ is widely connected, this is impossible. Hence $M$ is the sum of two mutually exclusive biconnected subsets $B_{11}$ and $B_{12}$.

In a similar manner for $n > 1$ it is seen that $M$ is the sum of $n+1$ mutually exclusive biconnected subsets $B_{n1}, B_{n2}, \cdots, B_{nn,n+1}$, where $B_{nj}$ contains $N_j + [V_{nj} + \sum_{a=1}^{10} M_{nja}] \times Q$ of $M$ and $B_{n1}$ contains all the rest of $M$, except the rest of $M$ contained in $Q_1$, and $B_{n2}$ contains this.

It is seen however that $M$ is not the sum of infinitely many mutually exclusive connected subsets $T_1, T_2, \cdots$, for every region of $Q$ contains a $J_d'$ and so each connected set $T_i$ would contain a $U_i$ dense in $V \cdot J_d'$ and so dense in $V \cdot Q$. This $U_i$ is also dense in $V$ because of the $J_{ia}$'s. Thus $T_1 \cdot V, T_2 \cdot V, \cdots$ is a division $D_i$ of $V$ into infinitely many mutually exclusive subsets $U_1, U_2, \cdots$ each dense in $V \cdot Q$. Hence one of these is a $U''$ which does not contain a point of some $J_d''$. Therefore the $T_i$, such that $U'' = T_i \cdot V$, cannot be connected.

Thus it is seen that $M$ is an example of a finitely-divisible connected set and similarly of a finitely-containing connected set, since each connected subset of $M$ is widely connected.

New Mexico State College

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7 E. W. Miller, loc. cit., p. 126, Theorem 7.